Local formalizations in nonlinear analysis and related areas and proof-theoretic tameness

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‘The object lesson concerns the passage from the foundational aims for which various branches of modern logic were originally developed to the discovery of areas and problems for which logical methods are effective tools. The main point stressed here is that this passage did not consist of successive refinements, a gradual evolution by adaptation as it were, but required radical changes of direction, to be compared to evolution by migration.’ ([45], p.139).

‘there is plenty of scope for specialist experience in logic provided (i) new questions are asked and (ii) that experience is combined with more specific knowledge.’ ([45], p.140).

1 Introduction

In the recent book [3], John Baldwin stresses that while early 20th century logic focused on the foundation of all of mathematics, ‘contemporary model theory makes formalization of *specific mathematical areas* a powerful tool’ and uses ‘local formalizations for distinct mathematical areas in order to organize and do mathematics, and to analyze mathematical practice’. Moreover, ‘geometry ... plays a fundamental role in analyzing the models of tame theories and solving problems in other areas of mathematics’ ([3], p.3). As a result of this ‘paradigm shift’, model-theoretic methods became a useful tool in core areas of mathematics such as algebra or algebraic geometry.

Baldwin cites [44] in this respect e.g. ‘Kreisel had identified one element of the malaise: “a preoccupation with a universal framework (a universal language, for example) and thus with logical possibilities. This preoccupation is at heart of the malaise; it concerns a potential conflict between pursuing these logical ideals and effective knowledge”([44])’ ([3], p.250). However, where Kreisel is skeptical about whether model-theoretic ‘transfer principles must take the literary form of metatheorems’ ([44] as quoted in [3] p.62), Baldwin states that ‘given the results discussed in this book, we see Kreisel as overly pessimistic about the prospects of metatheorems' ([3], p.62).

By its historical origin, proof theory has been particularly focused on general foundational issues such as the consistency strength of whole systems of analysis and set theory which extend basic number theory and so are not 'tame' in the model-theoretic sense. Rather than ruling out the possibility of 'Gödelian phenomena' by considering only tame theories one goal in recent decades has been to actually produce such phenomena in the context of ordinary mathematics. One famous example for this is Harvey Friedman's discovery that a finitary (Π^0_2-form) of Kruskal's theorem cannot be proven in predicative mathematics (in the sense of ATR_0). Many much stronger forms of such 'concrete incompleteness phenomena' have been discovered by Friedman during the last decades.

A different development, again initiated by Harvey Friedman [11] and then developed mainly by Stephen Simpson [56] and his collaborators under the name of 'reverse mathematics', has been to investigate the proof-theoretic strength of basic theorems used in core mathematics relative to a weak base system RCA_0 of second-order number theory which is Π^0_2-conservative over primitive recursive arithmetic PRA. One of the outcomes of this line of research has been that most of that part of existing ordinary mathematics (as long as it is formalizable in the language of 2nd order number theory) can be carried out already in systems such as ACA_0 or WKL_0 which are conservative over Peano arithmetic PA resp. the fragment of PA with Σ^0_1-induction (which in turn is Π^0_2-conservative over primitive recursive arithmetic PRA). This already indicates that a substantial amount of ordinary mathematics is tame in a proof-theoretic sense even when in principle non-tame structures such as a natural numbers are present.

While also reverse mathematics deals with **provability** in certain formal systems (though addressing theorems from ordinary mathematics), Kreisel asked already many decades ago for a shift of emphasis towards the 'unwinding' of **specific proofs**. Moreover, here the focus is on proofs of theorems A which have a **simpler logical form** than the noneffective set-theoretic tools used in the proof (and, in particular, are not equivalent to the latter in sense of reverse mathematics). In fact, the combinatorial or numerical nature of the theorems A considered naturally asks for explicit witnesses or bounds to be extracted from the prima facie noneffective proof. To use proof-theoretic methods for this new type of purpose has been proposed by Kreisel since the 50's, most specifically in his 'unwinding of proofs' program. This has been carried out under the name of 'proof mining' since around 2000 systematically in the area of nonlinear analysis such as fixed point theory, ergodic theory, abstract Cauchy problems, nonlinear semigroups, convex optimization and geodesic geometry (see [31] for an account of the vast influence Kreisel's insights have had on prompting this development and [23, 29] for surveys on the results obtained in the proof mining paradigm).

Many theorems in nonlinear analysis concern convergence results for iterated procedures (x_n) for the computation of fixed points of some mapping $T : C \to C$ (C typically being a convex subset of some normed or geodesic space), zeros or minimizers of mappings $F : C \to \overline{\mathbb{R}}$ etc.

Here one usually works in the context of abstract classes of normed and metric structures (used as parameters of the problem) which are not assumed to be separable and so cannot even be expressed in the language of 2nd order number theory. In fact, it is the very absence of any separability assumptions which makes it possible to obtain highly uniform bounds which only depend on general local metric bounds without any compactness assumptions required.

For so-called asymptotic regularity results $d(Tx_n, x_n) \to 0$ one typically obtains full rates of convergence which are mostly polynomial in the relevant data and moduli of the problem (see e.g. [30] for a polynomial rate of convergence which has been extracted from Bauschke's [4] solution of the 'zero displacement conjecture') or at least are simple exponential (as e.g. in [36], which analyzes an asymptotic regularity proof from [1] that uses iterated arithmetical comprehension, and [51]; see also below). For strong convergence theorems for (x_n) itself one usually can show that (unless for special cases where e.g. the uniqueness of the fixed point of T can be established or one has a so-called modulus of regularity) there is no computable rate of convergence even for simple cases such as $C := [0,1]$ and easily computable mappings T (see e.g. [50]). So one usually only has effective so-called rates $\Phi(\varepsilon, \underline{a}, g)$ of metastability (in the sense of Tao [57, 58])

$$\forall \varepsilon \in \mathbb{Q}^*_+ \, \forall g : \mathbb{N} \to \mathbb{N} \, \exists N \leq \Phi(\varepsilon, \underline{a}, g) \, \forall i, j \in [N, N + g(N)] \; (d(x_i, x_j) < \varepsilon)$$

which is an instance of Kreisel's no-counterexample interpretation applied to the Cauchy property (see [42, 43]). These are - with very few exceptions - of the form

$$(+) \; \Phi(\varepsilon, \underline{a}, g) = \Big(\chi_1(\varepsilon, \underline{a}) \circ \tilde{g} \circ \chi_2(\varepsilon, \underline{a})\Big)^{(B(\varepsilon, \underline{a}))}(0),$$

where χ_1, χ_2, B are simple (typically polynomial) functions in ε and (majorants of) the parameters $\underline{a}$ of the problem involved but which do not depend on g (here $\tilde{g}(n) := \max\{g(i) : i \leq n\} + n$) and $f^{(n)}(0)$ denotes the n-th iteration of f. The need for such an iteration usually is due to the arithmetical residuum of a use of sequential compactness which even in the case of a single use of the convergence of bounded monotone sequence of reals uses Σ^0_1-induction Σ^0_1-IA which suffices to show the totality of function iteration (see also below). In fact, the Cauchy property of monotone sequences in $[0,1]$ is equivalent to Σ^0_1-IA (see [18]).

So despite of the fact that convergence theorems of the form above crucially use quantification over natural numbers and so per se could display phenomena of enormous growth rates (as in Kruskal's theorem) it is an **empirical** fact, though, that with a few notable exceptions, proofs e.g. in analysis seem to be tame in the sense of allowing for the extraction of bounds of rather low complexity. It is this frequent **proof-theoretic tameness** in currently existing ordinary mathematics which makes the program of unwinding proofs so rewarding but which to diagnose requires a proof-theoretic analysis in each particular case.

Very recently, we analyzed a proof for a central strong convergence result in nonlinear analysis where - as it stands - the rate of metastability for the first time uses primitive recursion of type 1, i.e. the fragment T_1 of Gödel's T whose definable type-1 functions coincide with the provably total functions of the fragment of Peano arithmetic with Σ^0_2-induction (which is equivalent to Π^0_2-induction) and e.g. includes the Ackermann function. Only future research will show whether this is an artefact of the proof being analyzed (or whether even a closer examination of the extracted bound allows for a T_0-definition) or is best possible.

In each of the applications of proof mining it is the bound extracted and/or the general mathematical insights (also qualitative ones such as generalizations to geodesic settings and abstract versions, see e.g. [47]) into the mathematical situation at hand which is of interest rather than to add to the 'security' of the original proof: 'Of course, being special, finitist proofs do have some special properties including virtues. It just so happens that (special) reliability is not among them.' ([45], p.145). One of these virtues e.g. often is the fact that the analyzed proofs suggests immediate generalizations (e.g. to geodesic setting or more general classes of geodesic spaces etc., see e.g. [32, 36] or [46] which generalizes the analysis from [34] from CAT(0) to CAT(κ)-spaces for $\kappa > 0$).

2 General observations made in case studies I: extensionality

As mentioned already (and discussed in detail e.g. in [20, 12, 24]) most situations in nonlinear analysis involve abstract classes of normed or metric structures such as Hilbert spaces, uniformly convex spaces or CAT(κ) spaces which are determined by general geometric conditions while not assuming separability. The significance of the latter does not primarily rest on the greater generality but on the fact that if a proof does **not use separability** of the metric structures X the extracted **bounds are highly uniform**, i.e. independent from norm-bounded or metrically bounded parameters from X without any compactness assumption. This uniformity is of interest also in cases where the extracted bound is applied only to separable or even (boundedly) compact structures, e.g. by providing bounds which are independent of the dimension of the space under consideration.

In order to get the appropriate input data for the bounds, one may have to enrich X by suitable moduli e.g. of convexity or smoothness etc. Specially designed **logical metatheorems** which are **tailored** for the class of statements at hand guarantee for whole classes of proofs the extractability of uniform bounds which only depend on X via these moduli (compare Baldwin on **'local'** versus **'global'** formalizations in contemporary model theory). For the formalization of proofs in the context of abstract metric structures X we use systems formulated in the language of all finite types ρ over the base types $\mathbb{N}$ and X.

The structures admissible in these metatheorems must be axiomatizable by axioms which have a simple monotone functional interpretation (in the respective moduli). Usually, this is guaranteed by verifying that, given maybe some modulus $\omega : \mathbb{N} \to \mathbb{N}$, the axioms stating that X belongs to the respective class of structures (with modulus ω) can be expressed in purely universal form. This e.g. is the case for the following classes of spaces: metric, hyperbolic, CAT(0), CAT($\kappa > 0$), normed spaces, their completions, Hilbert, uniformly convex, uniformly smooth (not: separable, strictly convex or smooth) spaces, abstract L^p- and $C(K)$-spaces and all normed structures axiomatizable in positive bounded logic (Henson, Ben-Yaacov etc., see [13]).

Similarly, the conditions on the classes of admissible mappings T need to be axiomatizable in this way (again possibly using suitable moduli $\omega : \mathbb{N} \to \mathbb{N}$) which includes the following classes: uniformly continuous, Lipschitz-continuous, nonexpansive, firmly nonexpansive, strongly (quasi-)nonexpansive, pseudo-contractive mappings, directionally nonexpansive mappings, mappings satisfying Suzuki's condition (E), maximally monotone and accretive etc. mappings where in the latter two cases also set-valued operators $T : X \to 2^X$ have been treated.

Some of these conditions imply the uniform continuity of the operator T which then in turn implies the **extensionality** of T

$$(*)\ \forall x, y \in X \left(x =_X y \to T(x) =_X T(y)\right),$$

where $x =_X y :\equiv d_X(x, y) =_{\mathbb{R}} 0$. This extensionality must not be included as an axiom into the formal systems for which a metatheorem on uniform bound extractions can be expected to hold since, otherwise, the very statement of these metatheorems (on the extractability of uniform bounds) would imply the uniform continuity of T together with a modulus of uniform continuity (even largely independent of T). In fact, uniform continuity **is** the uniform quantitative version of extensionality. So when using conditions on T such as pseudo-contractivity, quasi- or directional nonexpansivity or Suzuki's condition (E) one cannot make free use of the extensionality axiom in formalizing a given proof. The principle $(*)$ (and a fortiori its extensions to higher types) appears to be the **only** principle used in mathematics which per se does not have any computational content. How then is this addressed in the practice of proof mining? Here roughly three different situations can occur:

1. The use of extensionality can be seen to be an instance of the (admissible in the aforementioned metatheorems) quantifier-free *rule* of extensionality

$$\frac{A_{qf} \to s =_X t}{A_{qf} \to T(s) =_X T(t)},$$

   where A_{qf} is a quantifier-free formula which may contain parameters.

2. There is an essential use of the axiom of extensionality but of a special form whose uniform quantitative version does not require a modulus of uniform continuity. A particularly important instance of this is the use of extensionality in the form

$$\forall x, y \in X \left(x =_X y \wedge Tx =_X x \to Ty =_X y\right)$$

   whose uniform quantitative form only requires so-called moduli of uniform closedness $\delta_T, \omega_T : \mathbb{N} \to \mathbb{N}$ such that

$$(*) \begin{cases} \forall x, y \in X\, \forall k \in \mathbb{N} \Big(d(x, y) < \frac{1}{\omega_T(k)+1} \wedge d(x, Tx) < \frac{1}{\delta_T(k)+1} \\ \qquad\qquad\qquad\qquad \to d(y, Ty) \le \frac{1}{k+1}\Big) \end{cases}$$

   which e.g. can easily be constructed if T satisfies Suzuki's condition (E) even though the latter does not imply the continuity of T (see [35, 29]).

3. If the extensionality axiom is needed but the properties of T do not imply the existence of a uniform bound on the resp. use of extensionality then such a modulus needs to be added as an assumption (a modulus of uniform continuity always suffices but weaker moduli as in the item above are often sufficient, see Proposition 4.15 and Remark 4.16 in [28]).

As is evident from the above, the proof-theoretic bound extraction methods do **not** generally require all functions involved to be uniformly continuous which is the common assumption in continuous or positive bounded logic (see, however, the recent paper [8]).

3 General observations made in case studies II: noneffective existence principles

The main noneffective existence principles applied in nonlinear analysis can be divided in the following two classes:

1. Principles which have the form of a so-called axiom

$$\Delta: \ \forall x^\delta \exists y \leq_\rho sx \forall z^\tau A_{qf}(x, y, z),$$

where s is some closed term of the system used and A_{qf} a quantifier-free formula. Here $u \leq_\rho v$ is pointwise defined, where if u has the type $\rho = \rho_1 \to (\ldots \to (\rho_k \to \tau)\ldots)$ with $\tau \in \{\mathbb{N}, X\}$ then v has the type $\rho_1 \to (\ldots \to (\rho_k \to \mathbb{N})\ldots)$ and $u(\underline{w}) \leq_X v(\underline{w}) :\equiv \|u(\underline{w})\| \leq v(\underline{w})$ in the normed case and $u(\underline{w}) \leq_X v(\underline{w}) :\equiv d_X(u(\underline{w}), a) \leq v(\underline{w})$ for some reference point $a \in X$ in the metric case (see [24] and [13]).

2. Principles such as (strong as well as weak) sequential compactness and the existence of projections (metric as well as sunny nonexpansive ones) as well as Banach limits which use arithmetical comprehension, sometimes prima facie also in its uniform version

$$(\exists^2): \ \exists \varphi \forall f^{\mathbb{N} \to \mathbb{N}} \left(\varphi(f) =_{\mathbb{N}} 0 \leftrightarrow \exists n^{\mathbb{N}} f(n) =_{\mathbb{N}} 0\right)$$

(see further below) or arithmetical dependent choice (and in the case of Banach limits even the existence of nontrivial ultrafilters).

If a proof uses sequential (weak or strong) compactness, usually one of the three following scenarios applies:

1. The use of sequential compactness (in the strongly compact case) can be replaced by Heine-Borel compactness and so can be reduced to the case of an axiom Δ either by using the binary König's lemma WKL or a - more easily applicable - nonstandard uniform boundedness principle Σ^0_1-UB which follows from a nonstandard axiom $F \in \Delta$ by means of quantifier-free choice (see [15, 17] and also [24], Chapter 12). 'Nonstandard' here refers to the fact that the resp. principles do not hold in the full set-theoretic model. Using a generalized uniform boundedness principle $\exists$-UB^X for the type X (see [22] and [24], 17.7.-17.8, for the bounded metric case, and [13]

for the normed case) the latter is sometimes even possible in cases where one uses the sequential **weak** compactness e.g. of bounded, closed and convex subsets C of an abstract Hilbert space. Then, however, there is no classically correct principle such as WKL which would imply the Heine-Borel compactness of C (since C is not Heine-Borel compact unless X is finite dimensional) but one **has** to use a strong non-standard uniform boundedness principle such as $\exists$-UB^X which, nevertheless, can be eliminated from the verification of the extracted bound and which does not contribute to the complexity of the bound (see Theorem 3.5 in [22] or Theorem 17.101 in [24]. This principle can also be seen as a version of the bounded collection principle used in the bounded functional interpretation of theories with abstract types (see [9] where bounded functional interpretation is used to establish proof-theoretic conservation results for bounded collection).

Uniform boundedness has been used implicitly (and subsequently eliminated) in the unwindings of proofs of theorems of Browder, Wittmann and Yamada (based on a sequential weak compactness argument) in [26] (for the theorems of Browder and Wittmann) and [40, 41] (for the theorem of Yamada). In both cases, the elimination of the sequential weak compactness argument leaves no contribution to the final bound at all which is of the simple primitive recursive form (+) above (the need for the primitive recursion does not come from the sequential compactness argument but from a projection argument discussed below). Recently, [10] made the hidden use of nonstandard uniform boundedness on which these proof minings were based explicit by formulating a general 'macro' which follows from uniform boundedness ($\exists$-UB^X) as well as from bounded collection (in the sense of [9]) and can be used to formalize (when adapted suitably to the situations at hand) proofs of the resp. theorems of Browder, Wittmann and a special case of Yamada's theorem due to Bauschke which no longer use sequential weak compactness. Then bounded functional interpretation is applied in [10] to the resulting proofs to extract bounds similar to those previously obtained in [26] and - in the more general context of Yamada's theorem - in [41].

Just as the use of sequential weak compactness in the original proof does not contribute at all to the final bounds extracted in [26, 41] (which are verified without that use), also the use of Banach limits in a proof analyzed in [34] in the end turned out to have no contribution to the extracted bound although, in general, such a use **could** contribute very significantly via the comprehension functional ($\exists^2$).

2. Compactness can be avoided altogether by making the original convergence proof constructive once the assumptions are appropriately uniformized. A typical instance of this is the recent unwinding ([37]) of a noneffective proof for a convergence theorem in the context of the classical Lion-Man game in [48]. Here, using a compactness assumption on a uniquely geodesic space satisfying the so-called betweenness property, it is shown by a nested use of sequential compactness that the lion eventually gets arbitrary close to the man (ε-capture). As it turns out, once the unique geodesic and the betweenness properties are upgraded to 'uniform' versions of these properties (with appropriate moduli so that these properties have the logical form admissible in the logical metatheorems) which - in the presence of compactness - are equivalent to the nonuniform ones, one can avoid the use of sequential compactness in the con-

vergence proof. Moreover, existing metatheorems guarantee the extractability of an explicit rate of convergence (depending on these moduli). This not only provides an effective quantitative version of the original convergence proof but also a vast generalization of the convergence statement itself since now instead of compactness only these uniformized properties need to be assumed. This e.g. applies to all bounded convex subsets of uniformly convex Banach spaces or of CAT(κ)-spaces (of sufficiently small diameter). and in both cases the respective moduli can easily be computed (and are low degree polynomials for L^p-spaces and CAT(κ)-spaces).

3. Sequential compactness is eliminated by arithmetizing the original proof. Already in [16] we showed that the use of (fixed sequences of instances of) sequential compactness in the form of the convergence principle for bounded monotone sequences of reals, the Bolzano-Weierstraß principle for bounded sequences in $\mathbb{R}^n$ and the Arzelà-Ascoli lemma can replaced by arithmetical principles (provable by Σ^0_1-induction which - by [18] - is optimal) in proofs of $\forall\exists$-statements if the deductive context does not allow for non-quantifier-free instances of induction which use the results of sequential compactness as parameters nor the iteration of the latter. Similarly, the use of (fixed sequences of instances of) the existence of the limsup of bounded sequences in $\mathbb{R}$ can be reduced to Π^0_2-induction (which - by [18] - again is optimal). The approach is based on an 'elimination of monotone Skolem functions' procedure which in [24] (17.9) is shown to apply also in the presence of abstract spaces X. This explains why in many proofs, despite of the use of sequential compactness, this 'arithmetization' of the proof results in primitive recursive bounds. E.g. this is the case in the elimination of a Bolzano-Weierstraß argument for abstract compact metric spaces in [21] and - much extended - in [35]. Here the use of compactness is not eliminated, in fact the bounds depend on a given modulus of total boundedness of the metric space, but the computational strength of its sequential form just causes a simple primitive recursive contribution to the rate of metastability which is of the form (+) discussed above. Note that in general already the full principle of monotone convergence for monotone sequences in $[0, 1]$ is equivalent to arithmetical comprehension ([56]) whose Gödel functional interpretation cannot be solved in Gödel's T (but requires bar recursion $B_{0,1}$ of lowest type; see [24]). Sometimes, the principle of monotone convergence cannot only be reduced to its primitive recursively bounded metastable version but - in the course of the analysis of a proof of a Π^0_2-theorem - to an instance of this metastable version with a very simple counterfunction g (e.g. a constant function $g \equiv k$). This is the reason why the analyses of proofs that originally used the convergence principle as carried out in [51] (Theorem 4.4) as well as in [36] (Theorem 3.1) resulted in simple exponential rates of convergence (in both cases the theorems in question state that a sequence of positive reals decreases towards 0 which is in Π^0_2).

Projection arguments are usually treated by first replacing them in a given proof by arithmetical ε-weakenings. E.g. the existence of the metric projection of $x \in X$ onto the (closed and convex) fixed point set $Fix(T)$ of a nonexpansive mapping T in a Hilbert space X, which is used in the aforementioned proofs of theorems of Browder, Wittmann and Yamada, can usually be replaced (see [26, 41] and - similarly - the recent [10]) by its arithmetical

version:
$$\forall \varepsilon > 0\, \exists y \in Fix(T)\, \forall z \in Fix(T)\ (\|x-y\| \le \|x-z\| + \varepsilon).$$

While the proof of the existence of the actual projection uses countable choice ([25]), the arithmetic version can be proved by induction and its quantitative version has a simple primitive recursive bound.

Browder's theorem (proved independently also by Halpern [14]) states that for a Hilbert space X, a bounded closed and convex subset $C \subseteq X$ and a nonexpansive mapping $T : C \to C$ the path (x_t) of resolvents

$$x_t = tTx_t + (1-t)x, \quad t \in (0,1), x \in C,$$

strongly converges for $t \to 1^-$ and, in fact, to the metric projection of x onto $Fix(T)$.

In the important paper [54], Browder's theorem was for the first time generalized from Hilbert spaces to more general Banach spaces X such as uniformly smooth spaces. Even for L^p-spaces (other than L^2) this was new. Unless X is a Hilbert space, the path (x_t) never converges to the metric projection of x but to the so-called sunny nonexpansive retraction of x onto $Fix(T)$. In general, nonexpansive retractions onto closed, bounded and convex sets C are known to exist only in special situations, e.g. when C is the fixed point set of a nonexpansive mapping as was shown by Bruck using Zorn's lemma ([5, 6, 7]). The only more 'constructive' approach to the existence of (unique sunny) nonexpansive retractions in this situation in fact stems from Reich's theorem. So here one cannot rely on a quantitative version of the existence of ε-versions of sunny nonexpansive retractions to analyze Reich's proofs but has to directly analyze the strong convergence of (x_t). This is done in [39] (where, in fact, a variant of Reich's proof due to [49] is analyzed). [49] uses the existence of infima of the function
$$F(y) := \limsup_{n\to\infty} \|x_{t_n} - y\|^2$$
where (t_n) is a sequence in $(0,1)$ which converges to 1. In [39] - using as an additional hypothesis that X is uniformly convex (in addition to being uniformly smooth, which still covers all L^p-spaces for $1 < p < \infty$) - a modulus of uniqueness for the infimum is constructed and used to replace the (unique) point where the infimum is attained by ε-infima. This in turn makes it possible to replace the existence of F as an object (which requires uniform arithmetical comprehension $(\exists^2)$) by

$$\forall y \in C\, \exists z \in \mathbb{R}\, (z = \limsup_{n\to\infty} \|x_{t_n} - y\|^2),$$

where '$z = \limsup_{n\to\infty} \|x_{t_n} - y\|^2$'$\in \Pi^0_3$, which only requires ordinary arithmetical comprehension.

Finally, in the whole proof even the use of limsup's is replaced by ε-limsup's whose existence is equivalent to Π^0_2-induction (that Π^0_2-IA suffices is straightforward; for the converse one has to adapt the proof of Theorem 6.1 in [18]). Hence - by [53] - the functional interpretation (combined with negative translation) of the existence of approximate limsup's can be carried out in the fragment T_1 of Gödel's T (which only has the primitive recursive recursors R_0 and R_1). Finally, the existence of of the resulting ε-infimum problem can be solved by functionals in T_2. A detailed analysis of the latter solution shows that in the concrete application at hand, the use of type-2 primitive recursion actually reduces to a type-1 primitive recursion resulting in a final rate of metastability for the Cauchy property of (x_{t_n}) which is definable in T_1 (see [39]). It seems likely that a further analysis of the (very complicated detailed structure of) this bound - in the line of Lemma 4 in [52] - might show that it actually is definable already in T_0. This then would leave the rate extracted for Baillon's nonlinear ergodic theorem in [27] (which is definable in T but - as it stands - not in T_0) as is the only bound extracted so far which is not primitive recursive in the ordinary sense of Kleene (i.e. in T_0). Among all the bounds extracted which are definable in T_0, the one obtained in [55] is the only one which does not have the simple form (+).

The situation discussed can be summarized as follows:

- While mostly uniform **classes** of metric and normed spaces X are used (as atoms) quantification over $\mathbb{N}$ is needed in all applications (already to speak e.g. about the convergence of sequences in X and rates of convergence or metastability), i.e. one does not have model-theoretic tameness and Gödelian or H. Friedman-type phenomena could occur in principle.
- **Empirical fact 1:** all of the rates of asymptotic regularity extracted so far are either polynomial or simple exponential in the basic data (for [19, 32] this holds for constant $\lambda_k = \lambda$ only). This also applies to the moduli of uniqueness extracted in best approximation theory (see [24], Chapter 16, for a survey).
- **Empirical fact 2:** among all the numerous rates of metastability extracted only 2 so far are not primitive recursive as it stands (but definable in Gödel's T and so $\alpha < \varepsilon_0$-recursive). With one exception, the primitive recursive bounds extracted all have a the simple form (+) discussed above. Usually, the primitive iteration involved in (+) can be shown to be necessary by establishing that the Cauchy property of the respective sequence (already in simple cases such as $C := [0, 1]$ and computable mappings $T : [0, 1] \to [0, 1]$) implies Σ^0_1-induction.
- In contrast to model-theoretic tameness, to **detect** proof-theoretic tameness **requires** to actually carry out a **proof analysis** in each individual case.
- **Geometric properties** such as uniform convexity and smoothness etc. are usually **more important than complicated inductions** (see, however, the discussion of [39] above). The proof-theoretic tameness of the axiomatization of these properties amounts to having a simple (if not trivial) monotone functional interpretation in suitable moduli $\omega : \mathbb{N} \to \mathbb{N}$ which quantitatively witness these properties. See e.g. [2] for converting prima facie noneffective proofs into explicit low-complexity transformations from certain moduli (e.g. of the uniform convexity of the given space) into others (e.g. of uniform continuity for the proximal mappings in uniformly convex spaces).

- The use of **uniform boundedness** (amplifying the already implicitly present uniformity in proofs and corresponding to the use of ultraproducts in continuous or positive bounded logic) does not contribute to the growth of extractable bounds.

- General proof-theoretic **logical metatheorems** play an important guiding role in finding promising applications.

Let us finally mention another important aspect of the proof-theoretic tameness of proofs in nonlinear analysis which has been observed over the past 20 years. Even if a proof does not use noneffective set-theoretic existence principles or complicated inductions, so that the extracted bounds are guaranteed to be of low complexity and even polynomials, the depth of the nesting of the basic functions, e.g. the degree of the polynomial, in general depends superexponentially on the quantifier-complexity of the formulas used in cuts (modus ponens). This already can happen in plain logic or logic augmented by purely universal axioms. Related to this, the bounds extracted by functional interpretation, which make use of the typed λ-calculus, could require superexponentially (in the degree of the highest type use) many β-reduction steps to compute their normal form. In practice, however, the normalization has never been a problem and usually is almost trivial (except for [39] where things are more involved) and the logical nestings of the basic functions are of very low depth. So not only is the use of mathematical principles typically tame proof-theoretically but even that of first-order logic.

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