Quantitative image recovery theorems

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Abstract: This paper provides a quantitative version of the classical image recovery problem to find an ϵ-approximate solution of the problem. The rate of asymptotic regularity of the iteration schemas, connected with the problem of image recovery, coincides with the existing optimal and quadratic bounds for Krasnoselskii-Mann iterations. We then provide explicit effective and uniform bounds on the approximate fixed points of the mappings under consideration to be an approximate solution of the image recovery problem up to a uniform change from ϵ to $\delta_{(\epsilon)}$. When combined, these results provide algorithms with explicit rates of convergence for the recovery of an ϵ-perturbation of the original image in different settings.

1. Introduction and Preliminaries

In 1965, Browder-Göhde-Kirk proposed, independently, the theory of nonexpansive mappings in uniformly convex Banach spaces (see e.g. [11, 12]). Since then, the fixed point theory for nonexpansive mappings is an active area of research in nonlinear functional analysis and found a diverse range of applications, for instance problems of zeros of a monotone operator and variational inequality problems. The problem of finding a common fixed point of a finite family of nonlinear mappings acting on a nonempty convex domain often arises in applied mathematics. For example, finding a common fixed point of a finite family of nonexpansive mappings may be used to solve systems of simultaneous equations, convex minimization problems of functions and the problem of image recovery. The latter problem has been analyzed in Hilbert spaces and further generalized to uniformly convex Banach spaces and found many useful applications in applied mathematics, for instance partial differential equations, control theory and image and signal reconstruction. The purpose of this paper is to analyze different iteration schemas, involving a finite family of nonlinear mappings, which are closely related to the problem of image recovery.

Let H be a real Hilbert space and let $C_1, C_2, \cdots, C_r$ be nonempty closed convex subsets of H. The problem of image recovery in a real Hilbert space H is defined as follows:
The original (unknown) image z is known a priori to belong to the intersection $C_0 = \cap_{i=1}^r C_i$ of the closed convex sets $C_1, C_2, \cdots, C_r$. An iteration schema involving the metric projections $P_i : H \to C_i$ onto the corresponding sets C_i, in which some initial estimate

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is sequentially projected onto the individual sets according to a periodic schedule, recovers some $z \in C_0$. The image recovery problem is also studied under the label 'convex feasibility problem' (see e.g. [3]).

One of the most important notions in metric fixed point theory is the asymptotic regularity [5] of a nonlinear iteration $\{x_n\}$ under consideration (see e.g. [2]). A Picard iteration of the nonlinear mapping $T : C \to C$ is said to be asymptotically regular if

$$\lim_{n \to \infty} \left\| T^n x - T^{n+1} x \right\| = 0. \tag{1.1}$$

Asymptotic regularity is not only useful in proving that fixed points exist but also in showing that the sequence of iterates $\{x_n\}$ converges (at least weakly) to a fixed point. In the theory of image recovery the following form of asymptotic regularity is essentially due to Crombez [8]:

Theorem 1.1 ([8, Theorem 2]). *Let $T : H \to H$ be a mapping given by*

$$T = \alpha_0 I + \sum_{i=1}^{r} \alpha_i T_i, \ \ 0 < \alpha_i < 1 \ \forall \ i = 0, 1, 2 \cdots, r \ , \ \sum_{i=0}^{r} \alpha_i = 1 \tag{1.2}$$

where

    (i) *each T_i is nonexpansive on H;*
    (ii) *the set of fixed points of T is nonempty;*
    (iii) *$Tu = u \Leftrightarrow T_i u = u \ \forall \ i = 0, 1, 2 \cdots, r$.*

Then T is asymptotically regular.

Conditions (ii) and (iii) can be summarized as $F(T) = \cap_{i=1}^{r} F(T_i) \neq \emptyset$. However, an inspection of the proof in [8] shows that actually only $\cap_{i=1}^{r} F(T_i) \neq \emptyset$ is needed for the asymptotic regularity.

This result has been generalized to uniformly convex Banach spaces:

Theorem 1.2 ([28, Theorem 5.4.2]). *Let X be a uniformly convex Banach space with modulus of uniform convexity η and let C be a nonempty convex subset of X. Let $T : C \to C$ be a mapping as defined in (1.2) where each T_i is nonexpansive on C and $\cap_{i=1}^{r} F(T_i) \neq \emptyset$. Then T is asymptotically regular.*

In the context of the image recovery problem, the mappings T_i are defined as

$$T_i := I + \lambda_i (P_i - I), \tag{1.3}$$

where the $P_i : C \to C_i$ are

    (i) metric projections in the case of Hilbert spaces H, $C := H$ and $0 < \lambda_i < 2$ for all i,
    (ii) nonexpansive retractions in uniformly convex spaces, where then, however, one has to restrict the coefficients to $0 < \lambda_i < 1$ for all i (nonexpansive retracts and retractions are e.g. discussed in [6, 25, 21]).

In 1992, Crombez [9] introduced and analyzed another parallel computing iteration schema by considering a mapping T as a convex combination solely of the mappings T_i defined in (1.3). That is

$$T = \sum_{i=1}^{r} \alpha_i T_i, \ \forall \ i = 1, 2 \cdots, r; \ \alpha_i > 0 \ \text{ and } \sum_{i=1}^{r} \alpha_i = 1. \tag{1.4}$$

Iterates of such mappings T have been studied first in [26]. The Picard iteration of the above mapping T is asymptotically regular and exhibits weak convergence in Hilbert spaces.

Metric projections are an essential ingredient of the iteration schema used for the recovery of the image. Since the construction of T_i involves the metric projection P_i which is characterized as a nonexpansive mapping in Hilbert spaces also the relaxed metric projection T_i defined in (1.3) is nonexpansive. Furthermore, the set of common fixed points of P_i coincides with that of T_i. Crombez ([8, 9]) shows that the set of fixed points of T coincides with $C_0 = \cap_{i=1}^r C_i$, where T is defined as in (1.2) or as in (1.4). From this he gets that $(T^n(x))$ weakly converges to a point $p \in C_0 = \cap_{i=1}^r C_i = \cap_{i=1}^r F(T_i)$.

However, the classical image recovery problem lacks any information on how a δ-fixed point of T relates to being in the intersection $C_{0,\epsilon}$ of ϵ-neighborhoods $C_{i,\epsilon}$ of C_i. Moreover, the problem of image recovery is often and seriously dealt with the inconsistent constraints i.e., when the intersection of the sets $C_1, C_2, \cdots, C_r$ is empty (see e.g. [13, 7]). To answer this question but also to get explicit effective rates of convergence we now introduce an ϵ-version of the classical image recovery problem which we will solve with explicit bounds in this paper. Our proposed ϵ-version provides an approximate solution of the problem even in a situation when constraints are inconsistent.

Let H be a real Hilbert space and let $C_1, C_2, \cdots, C_r$ be nonempty closed convex subsets of H. Let $\epsilon > 0$ and let $C_{1,\epsilon}, C_{2,\epsilon}, \cdots, C_{r,\epsilon}$ be the corresponding 'r' nonempty ϵ-convex subsets of H, where $C_{i,\epsilon} := \cup_{x \in C_i} B_\epsilon(x)$ (for $1 \le i \le r$) and define $C_{0,\epsilon} := \cap_{i=1}^r C_{i,\epsilon}$. Here $B_\epsilon(x)$ is the open ϵ-ball around x. The ϵ-version of the problem of image recovery in a real Hilbert space H is defined as follows:

The original (unknown) image z is known a priori to belong to the intersection $C_{0,\epsilon} = \cap_{i=1}^r C_{i,\epsilon}$ of the convex sets $C_{1,\epsilon}, C_{2,\epsilon}, \cdots, C_{r,\epsilon}$. For this to happen z does not have to be a fixed point of T but only a $\delta_{(\epsilon)}$-fixed point (for a suitable $\delta_{(\epsilon)}$ which does not depend on z) and instead of $C_0 \ne \emptyset$ it will turn out to be sufficient that $C_{0,\delta_{(\epsilon)}} \ne \emptyset$. Given an explicit rate of asymptotic regularity for our iteration schemas $x_n := T^n x$ involving the metric projections P_i onto the corresponding closed convex sets C_i we can get an explicit bound $\Psi(\epsilon)$ such that

$$\forall n \ge \Psi(\epsilon)\, (x_n \in C_{0,\epsilon}).$$

In addition to $\epsilon > 0$, Ψ only depends (in the case of T defined by (1.4)) on an $N \in \mathbb{N}$ such that $1/N \le \min\{\alpha_i \lambda_i, 2 - \lambda_i : 1 \le i \le r\}$ an upper bound $d \ge \|x_0 - p\|$ for some fixed point p of T and an upper bound $D > dist(x, C_{0,\delta_{(\epsilon)}})$.
For $0 < \lambda_i < 1$ the same type of result holds in arbitrary convex subsets $C \subseteq X$ of uniformly convex Banach spaces (with $C_1, \ldots, C_r \subseteq C$ closed and convex) where now $P_i : C \to C_i$ can be an arbitrary nonexpansive retraction. Then N has to satisfy $1/N \le \min\{\alpha_i \lambda_i (1 - \lambda_i) : 1 \le i \le r\}$ and Ψ additionally depends on some modulus of uniform convexity η of X.

Let us briefly indicate that the extractability of a uniform $\delta_{(\epsilon)}$ from the proof that

$$(1)\ C_0 \ne \emptyset \to F(T) \subseteq C_0$$

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is an instance of a general logical 'metatheorem' in the sense of [18, 10, 19]: (1) can be written as (using that trivially $x \in C_i \leftrightarrow P_i x = x$)

$$(2) \; \exists p \bigwedge_{i=1}^{r} (P_i p = p) \to \forall x \, (Tx = x \to \bigwedge_{i=1}^{r} (P_i x = x)),$$

where here p, x range over H in the Hilbert space case resp. over C in the uniformly convex case. (2) can logically be reformulated as

$$(3) \; \forall x \forall p \forall k \in \mathbb{N} \, \exists n \in \mathbb{N} \, \big(\bigwedge_{i=1}^{r} (\|P_i p - p\| \leq 2^{-n} \wedge \|Tx - x\| \leq 2^{-n} \to \bigwedge_{i=1}^{r} (\|P_i x - x\| < 2^{-k})).$$

Since Hilbert (as well as uniformly convex) spaces and abstract closed convex subsets C_i are allowed in the aforementioned logical metatheorems and metric projections resp. nonexpansive retractions can be axiomatized by purely universal axioms (see [14] for the former case) and are trivially majorized due to their nonexpansivity, (3) is of the right form so that the general logical metatheorems allow for the extraction of some uniform bound $\Phi(k, D, N)$ on $\exists n \in \mathbb{N}$ which only depends on the error 2^{-k}, an upper bound $D \geq \|x - p\|$ and N as discussed above (and in the uniformly convex case also on a modulus η (in principle one might also need $D \geq \|x\|$ in the normed case but this can be avoided here since the whole argument only takes place in convex subsets where only relative distances matter, see [19]). Since

$$(4) \; \|P_i x - x\| < 2^{-k} \to x \in C_{i, 2^{-k}}$$

(because of $P_i x \in C_i$) and (using that the P_i are nonexpansive retractions)

$$(5) \; p \in C_{i, 2^{-k-1}} \to \|P_i p - p\| < 2^{-k}$$

(because of $\|p - y\| < 2^{-k-1} \to \|P_i p - p\| \leq \|P_i p - P_i y\| + \|y - p\| \leq 2\|y - p\| < 2^{-k}$ for $y \in C_i$ with $\|p - y\| < 2^{-k-1}$) this yields that for all $x \in H, k \in \mathbb{N}$ and $\Phi'(k) := \Phi(k, D, N) + 1$

$$\|Tx - x\| \leq 2^{-\Phi'(k)} \wedge C_{0, 2^{-\Phi'(k)}} \neq \emptyset \wedge D > dist(x, C_{0, 2^{-\Phi'(k)}}) \to x \in C_{0, 2^{-k}}\big)$$

in the Hilbert case (see Theorem 3.1) and - with $x \in C$ and $C \cap C_{0, 2^{-\Phi'(k)}}$ instead of $C_{0, 2^{-\Phi'(k)}}$ - in the uniformly convex case with general C (see Theorem 3.4).
Note that (5) implies that

$$(6) \; \forall x \, \big(x \in C_0 \leftrightarrow \forall \epsilon > 0 (x \in C_{0, \epsilon}) \big),$$

which further supports that the concept $C_{0, \epsilon}$ is a natural one.

2. Rates of asymptotic regularity

As mentioned already, the asymptotic regularity of the iterations (1.2) and (1.4) (with T_i being defined as in (1.3)) constitutes a major part of the corresponding weak convergence results used to solve the problem of image recovery. It is, therefore, relevant to compute explicit and effective rates of asymptotic regularity.
For this it is sufficient to observe that the iterates of the mapping T defined in (1.2) coincide with the Krasnoselskii iteration for constant α_0 and, consequently, one can use the known optimal quadratic bound from [1] (in the case of Hilbert space, such a bound follows much more easily as a special case from [17]) as bound on the rate of asymptotic regularity of Picard iteration of the mapping T defined in (1.2). Krasnoselskii-Mann

iterations are studied - also for fixed point free mappings - in [4] (see [16] and [20] for quantitative versions of [4]). For $r = 1$, obviously the iterates of the mapping T defined in (1.2) reduce to the usual Krasnoselskii iteration for constant α_0. However, the same is true for general r as was first noticed in [30]:

Let $S := \sum_{i=1}^{r} \beta_i T_i$, where $\beta_i = \frac{\alpha_i}{1-\alpha_0}$ so that $0 < \beta_i < 1$ and $\sum_{i=1}^{r} \beta_i = 1$. The nonexpansivity of $T_1, T_2, \cdots, T_r$ implies the nonexpansivity of the convex combination $S : C \to C$. Hence (1.2) reduces to the Krasnoselskii iteration:

$$T := \alpha_0 I + (1 - \alpha_0) S.$$

Now we can apply the optimal rate of convergence of $\|x_n - Sx_n\| \to 0$ due to Baillon and Bruck [1] $\tilde{\Phi}(d, \epsilon, \alpha_0) := \frac{d^2}{\pi \alpha_0 (1-\alpha_0) \epsilon^2}$ which holds in any normed space as long as the sequence $(\|x_0 - Sx_n\|)_n$ bounded (say by d) which e.g. is the case if $p \in F(T) = F(S)$ and $d/2 \geq \|x_0 - p\|$ since

$$\|x_0 - Sx_n\| \leq \|x_0 - p\| + \|p - Sx_n\| \leq \|x_0 - p\| + \|p - x_n\| \leq 2\|x_0 - p\|.$$

So we do not need any common fixed point of $T_1, \ldots, T_r$. In fact, we not even need a fixed point of T as it is sufficient to assume that T has arbitrarily good approximate fixed points that are $d/2$-close to x_0. Since $\|x_n - x_{n+1}\| = (1 - \alpha_0)\|x_n - S(x_n)\|$ we get $\tilde{\Phi}(d, \epsilon/(1 - \alpha_0), \alpha_0) = \frac{(1-\alpha_0)d^2}{\pi \alpha_0 \epsilon^2} \leq \frac{d^2}{\pi \alpha_0 \epsilon^2}$ as rate of asymptotic regularity for T. Finally, Theorem 1.2 holds in arbitrary normed spaces (and - suitably adapted - even in any W-hyperbolic space in the sense of [19], where we then have to use the exponential rate of asymptotic regularity from [20]). W-hyperbolic spaces are closely related to the spaces of hyperbolic type in [11] and the hyperbolic spaces in the sense of [27] (see [19] for a detailed discussion).

Let us summarize things (for the normed case) in the following theorem:

Theorem 2.1. *Let X be a normed linear space and let C be a nonempty convex subset of X. Let $T : C \to C$ be a mapping as defined in (1.2) where each $T_i : C \to C$ is nonexpansive. Then T is asymptotically regular, whenever $d \geq \|x - p\|$ for $x \in C$ and $p \in F(T)$ (in fact it suffices to have arbitrarily good approximate fixed points of T that are d-close to x). In this case we have the following quantitative result:*

$$\forall \epsilon > 0 \, \forall n \geq \Phi(2d, \epsilon, N) \left(\|T^n x - T^{n+1} x\| \leq \epsilon \right),$$

where $N \in \mathbb{N}$ with $1/N \leq \alpha_0 \leq 1$ and $\Phi(d, \epsilon, N) := \frac{d^2 \cdot N}{\pi \epsilon^2}$.

It has been shown in [9] that the mapping (1.4) can be reduced to the one defined in (1.2) with a slight modification in the coefficients α_i and λ_i. We include the details of this since we need some quantitative estimates on the new coefficients later:
Let $0 < \alpha_i < 1$ for $1 \leq i \leq r$. In the following, it suffices that at least one of the λ_i is strictly less than 2. Without loss of any generality, we assume that $0 < \lambda_1 < 2$ and $0 < \lambda_i \leq 2$ for $2 \leq i \leq r$. Let $N \in \mathbb{N}$ be such that

$$\frac{1}{N} \leq 2 - \lambda_1 \quad \text{and} \quad \frac{1}{N} \leq \alpha_1. \tag{2.1}$$

Let $K := 2N + 1$. Then $K \geq \frac{\lambda_1}{2-\lambda_1} + 1$ (utilizing (2.1)). We now define new coefficients $\lambda_1' := \frac{K}{K-1}\lambda_1$, so that $0 < \lambda_1' \leq 2$, and $\beta_1 := \frac{K-1}{K}\alpha_1$. Then obviously $0 < \beta_1 < \alpha_1 < 1$

and $\beta_1 \lambda_1' = \alpha_1 \lambda_1$. Now define

$$
\begin{aligned}
\beta_0 \quad : \quad &= \alpha_1 - \beta_1 \\
&= \alpha_1 - \frac{K-1}{K}\alpha_1 \\
&= \frac{1}{K}\alpha_1.
\end{aligned}
$$

Put together, (1.4) now takes the form

$$
T = \beta_0 I + \sum_{i=1}^{r} \beta_i T_i', \tag{2.2}
$$

where

$$
\beta_0 := \frac{\alpha_1}{K}, \quad \beta_1 := \frac{K-1}{K}\alpha_1 \quad \text{and} \quad \beta_i = \alpha_i \quad \text{for} \quad 2 \leq i \leq r,
$$

$$
T_1' := I + \lambda_1'(P_1 - I), \quad \lambda_1' := \frac{K\lambda_1}{K-1},
$$

$$
\lambda_i' = \lambda_i \quad \text{and} \quad T_i' = T_i = I + \lambda_i(P_i - I) \quad \text{for} \quad 2 \leq i \leq r.
$$

Note that $\beta_i \in (0,1)$ and $\sum_{i=0}^{r} \beta_i = 1$.

Observe that, by (2.2), T has the form defined in (1.2) and consequently the iterates of the mapping T coincide with the Krasnoselskii iteration of the form:

$$
T = \beta_0 I + (1 - \beta_0)S,
$$

where S is defined as before, namely as

$$
S := \sum_{i=1}^{r} \gamma_i T_i', \quad \text{with} \quad \gamma_i := \frac{\beta_i}{1 - \beta_0}.
$$

Note that S is nonexpansive provided that the T_i' are. Hence one can use the optimal quadratic bound from [1] as a bound on the rate of asymptotic regularity for the iterates of (2.2).

Utilizing (2.1), we compute a lower bound of $\beta_0 (1 - \beta_0)$ as follows: Since $\frac{1}{N} \leq \alpha_1$, we get $\beta_0 \geq \frac{1}{NK} = \frac{1}{N(2N+1)}$. Moreover

$$
\begin{aligned}
1 - \beta_0 \quad &= \quad 1 - \frac{\alpha_1}{K} \\
&\geq \quad 1 - \frac{1}{K} \quad \text{(since } \alpha_1 < 1) \\
&= \quad \frac{K-1}{K} \\
&= \quad \frac{2N}{2N+1}.
\end{aligned}
$$

From the above estimates, we conclude the lower bound to be $\frac{1}{(2N+1)^2} \leq \beta_0 (1 - \beta_0)$.

Since $\beta_0 \in (0,1)$ we have that $F(T) = F(S)$.

In order to compute a rate of asymptotic regularity for the iteration schema (1.4), we have to assume the nonexpansivity of T_i defined in (1.3). In a Hilbert space, the nonexpansivity of T_i (as well as of T_i') follows from the following result:

Lemma 2.2 ([8])**.** *For $0 \leq \lambda_i \leq 2$, T_i is nonexpansive.*

Taking into consideration these facts we get a result, parallel to Theorem 2.1, on the rate of asymptotic regularity of iteration (1.4).

Theorem 2.3. *Let H be a Hilbert space and let $T : H \to H$ be the mapping defined by (1.4) with $T_i := I + \lambda_i(P_i - I)$, $0 \le \lambda_i \le 2$, $\lambda_1 < 2$, where $P_i : H \to C_i$ is a metric projection of H onto some closed convex subset $C_i \subseteq H$. Let $x_n = T^n x_0$ for $x_0 \in H$ and let $\|x_0 - p\| \le d > 0$ for some $p \in F(T)$. Then, we have*

$$\forall \epsilon \in (0,2] \, \forall n \ge \Phi\left(2d, \epsilon, (2N+1)^2\right) \ (\|x_n - x_{n+1}\| \le \epsilon),$$

where Φ is defined in Theorem 2.1 and $N \in \mathbb{N}$ such that $1/N \le \min\{\alpha_1, 2 - \lambda_1\}$.

Remark 2.4. *It is clear that it suffices that some $\lambda_j < 2$ and that $1/N \le \min\{\alpha_i, 2 - \lambda_j\}$ for some i, j.*

It is remarked that the mapping T defined in (1.4) is further analyzed by Takahashi and Tamura [30] in uniformly convex Banach spaces provided that $0 < \lambda_i < 1$ for $1 \le i \le r$. Observe that this particular choice of λ_i leads to the nonexpansivity of T_i being a λ_i-convex combination of I and P_i. In this situation, P_i could be any nonexpansive self-mapping of some convex subset $C \subseteq X$. We again have to re-define $\lambda_1' := \frac{K}{K-1}\lambda_1$ for $K := N + 1 \ge \frac{\lambda_1}{1-\lambda_1} + 1$ (with N such that $1/N \le \min\{1 - \lambda_1, \alpha_1\}$) in order to get that $\lambda_1' \le 1$ which is needed for the nonexpansivity of T_1'. Then we get the lower bound $\frac{1}{(N+1)^2} \le \beta_0(1 - \beta_0) \le \beta_0$ this time. Hence, we arrived at the following result.

Theorem 2.5. *Let C be a nonempty convex subset of a normed linear space X and let $T : C \to C$ be the mapping defined by (1.4) with $T_i := I + \lambda_i(P_i - I)$, $0 < \lambda_i < 1$ where $P_i : C \to C$ is a nonexpansive mapping. Let $x_n = T^n x_0$ for $x_0 \in C$ and let $\|x_0 - p\| \le d > 0$ for some $p \in F(T)$. Then, we have*

$$\forall \epsilon \in (0,2] \, \forall n \ge \Phi\left(2d, \epsilon, (N+1)^2\right) \ (\|x_n - x_{n+1}\| \le \epsilon),$$

where Φ is defined in Theorem 2.1 and $N \in \mathbb{N}$ such that $1/N \le \min\{\alpha_1, 1 - \lambda_1\}$.

Remark 2.6. Quite recently, a rate of asymptotic regularity for the Picard iterates of the mapping T defined in (1.4) has been computed in the general setting of uniformly convex hyperbolic spaces [24, Theorem 5.4]. Compared to [24, Theorem 5.4] our bound in Theorem 2.5 holds in an arbitrary normed linear space and does not depend on a modulus of uniform convexity η. Moreover, Theorem 2.5 can be generalized to uniformly convex hyperbolic spaces or to arbitrary W-hyperbolic spaces provided that T is defined as $T := \beta_0 I + (1 - \beta_0)S$ with S as before (for constant $\beta_0 := \frac{\alpha_1}{K}$ one can use the known bounds from [20] and [23], respectively). However, it is unclear whether this definition coincides with the one corresponding to (1.4) if one is not in a linear setting.

We conclude this section with a somewhat different but related quantitative asymptotic regularity result: recently, Khan and Kohlenbach [15] extracted uniform bounds on the asymptotic regularity of an iteration involving a finite family of nonexpansive mappings due to Kuhfittig [22]. That bound was recursive in nature due to the cyclic nature of the iteration as well as the inter-dependence of the asymptotic regularity of each mapping of the family. The bounds in [15] depend only on an upper bound on the distance between the starting point and some common fixed point, a lower bound $1/N \le \alpha_0(1 - \alpha_0)$, the error $\epsilon > 0$ and a modulus η of uniform convexity. As in [15] this uniformity of the

bound can be explained in terms of a general logical metatheorem from [10, 18] (see also [19]).

In 2000, Takahashi and Shimoji [29] introduced an iteration, namely S-mapping, involving a finite family of nonexpansive mappings. In a uniformly convex Banach space, the iteration is weakly convergent to a common fixed point of the family and capable of solving the problem of image recovery.

Let C be a convex subset of a Banach space X and let $T_1, T_2, \cdots, T_r$ be finite mappings of C into itself and let $\alpha_1, \alpha_2, \cdots, \alpha_r$ be real numbers such that $0 < \alpha_i < 1$ for every $i = 1, 2, \cdots, r$. Then, we define S-mapping of C onto itself as follows:

$$
\begin{aligned}
U_1 &= \alpha_1 T_1 + (1 - \alpha_1) I \\
U_2 &= \alpha_2 T_2 U_1 + (1 - \alpha_2) I \\
&\vdots \\
U_{r-1} &= \alpha_{r-1} T_{r-1} U_{r-2} + (1 - \alpha_{r-1}) I \\
S = U_r &= \alpha_r T_r U_{r-1} + (1 - \alpha_r) I.
\end{aligned}
\tag{2.3}
$$

Such a mapping S is called S-mapping generated by $T_1, T_2, \cdots, T_r$ and $\alpha_1, \alpha_2, \cdots, \alpha_r$. It is remarked that the iteration schema (2.3) is slightly more general than the classical Kuhfittig iteration (which is the special case where $\alpha_1 = \ldots = \alpha_r$). Quite recently, the classical Kuhfittig iteration has been analyzed in [15] in great detail covering various possible modifications and extracting corresponding bounds on the asymptotic regularity in uniformly convex W-hyperbolic spaces. Hence, bounds on the iteration (2.3) can easily be derived from [15, Theorem 3.2] with a slight modification. Taking into account these facts, we have the following result regarding bounds on asymptotic regularity of the iteration schema (2.3) in uniformly convex W-hyperbolic spaces (in the sense of [23]).

Theorem 2.7. *Let C be a nonempty convex subset of a uniformly convex W-hyperbolic space with monotone modulus of uniform convexity η and let $\{T_i\}_{i=1}^r$ be a finite family of nonexpansive self-mappings of C with $\cap_{i=1}^r F(T_i) \neq \emptyset$. Let $p \in \cap_{i=1}^r F(T_i)$ be such that $d(x_0, p) \leq D > 0$ for $x_0 \in C$ and let $\mathbb{N} \ni N > 0$ be such that $\frac{1}{N} \leq \min\{\alpha_i (1 - \alpha_i) : 1 \leq i \leq r\}$. Then for the sequence $x_n := S^n x_0$ generated by (2.3), we have*

$$
\forall \epsilon \in (0, 2] \, \forall n \geq \Phi_i(D, \epsilon, N, \eta) \ (d(x_n, T_i x_n) \leq \epsilon) \ \text{for} \ 1 \leq i \leq r
$$

where

$$
\Phi_i := \theta\left(\widehat{\eta}^{(r-i+\min(1, r-1))}\left(\frac{\epsilon}{2}\right)\right);
$$

with

$$
\theta(\epsilon) := \left\lceil \frac{D}{\widehat{\eta}(\epsilon)} \right\rceil;
$$

$$
\widehat{\eta}(\epsilon) := \frac{1}{N} \eta\left(D, \frac{\epsilon}{D+1}\right) \epsilon.
$$

Hint for the proof: Let $N > 0$ and set $\frac{1}{N} := \min\{\alpha_i(1 - \alpha_i) : 1 \leq i \leq r\}$. Now one can replace $\lambda(1 - \lambda)$ in the original proof of [15, Theorem 3.2] (for example, in the computation of estimates (3.4), (3.9) and the consequent $\widehat{\eta}(\epsilon)$) by $\frac{1}{N}$.

3. Approximate Fixed Points as Approximate Image Points

In this section, we show how the set of approximate fixed points of the mapping T corresponds (uniformly) to the points in $C_{0,\epsilon}$ up to a uniform change from ϵ to $\delta_{(\epsilon)}$.

Recall that a metric projection P onto the convex subset C of a real Hilbert space H is characterized by the following well-known fundamental inequality (see e.g. Proposition 3.5(d) in [12]):

$$\langle x - Px, y - Px \rangle \leq 0 \quad \text{for any } x \in H \text{ and for all } y \in C.$$

Our first main result of this section is a quantitative version of Proposition 3.1 in [9] which has been obtained by analyzing the proof in [9]:

Theorem 3.1. *(i) Let H be a Hilbert space and let $T : H \to H$ be a mapping defined as $T := \sum_{i=1}^{r} \alpha_i T_i$, $0 < \alpha_i < 1, \sum_{i=1}^{r} \alpha_i = 1$ with $T_i := I + \lambda_i(P_i - I)$, $0 < \lambda_i \leq 2$ where $P_i : H \to C_i$ is a metric projection of H onto some closed convex subset $C_i \subseteq H$. For every $\epsilon \in (0,1]$ and $D \in \mathbb{N}$ there exists a $\delta_{(\epsilon)} > 0$ such that*

$$\forall x \in H \ \left(\|Tx - x\| \leq \delta_{(\epsilon)} \wedge C_{0,\delta_{(\epsilon)}} \neq \emptyset \wedge D > dist\left(x, C_{0,\delta_{(\epsilon)}}\right) \Longrightarrow x \in C_{0,\epsilon}\right),$$

where

$$\delta_{(\epsilon)} := \frac{\epsilon^2}{22N^3 (D+1)(4N+1)},$$

and $N \in \mathbb{N}$ with $1/N \leq \min\{\alpha_i\lambda_i : 1 \leq i \leq r\}$.
(ii) The same result holds for $T := \alpha_0 I + \sum_{i=1}^{r} \alpha_i T_i$ for $0 < \alpha_i < 1$ with $\sum_{i=0}^{r} \alpha_i = 1$ and T_i as before and $1/N \leq \min\{\alpha_i\lambda_i : 1 \leq i \leq r\}$ (note that we do not need any positive lower bound on α_0).

Proof. (i) Let $a_i := \alpha_i\lambda_i \in (0,2)$. First note that

$$2N \geq \sum_{i=1}^{r} \frac{a_i}{a_r} \geq \sum_{i=1}^{r-1} \frac{a_i}{a_r} \tag{3.1}$$

since $N \geq 1/a_r$ and $\sum_{i=1}^{r} a_i \leq 2$ by the assumption on N, α_i, λ_i.
Let $\epsilon \in (0,1]$ and $\|Tx - x\| \leq \delta_{(\epsilon)}$ for some $x \in H$, where $\delta_{(\epsilon)} > 0$. We aim to show that $x \in C_{0,\epsilon}$. Let $y \in C_{0,\delta_{(\epsilon)}}$ be such that $\|x - y\| \leq D$. Since P_i is the metric projection onto C_i we get $\|P_i y - y\| \leq \delta_{(\epsilon)}$. Moreover

$$\begin{aligned}
\|P_i x - x\| &\leq \|P_i x - P_i y\| + \|P_i y - y\| + \|y - x\| \\
&\leq 2\|x - y\| + \delta_{(\epsilon)} \\
&\leq 2(D+1). \tag{3.2}
\end{aligned}$$

Now observe that

$$\|Tx - x\| = \left\|\sum_{i=1}^{r} a_i (P_i x - x)\right\| = \left\|\sum_{i=1}^{r-1} a_i (P_i x - x) - a_r(x - P_r x)\right\| \leq \delta_{(\epsilon)} \tag{3.3}$$

and hence

$$\left\|\sum_{i=1}^{r-1} \frac{a_i}{a_r}(P_i x - x) - (x - P_r x)\right\| \leq \frac{\delta_{(\epsilon)}}{a_r}. \tag{3.4}$$

Let $e := \sum_{i=1}^{r-1} \frac{a_i}{a_r} (P_i x - x)$ and $f := x - P_r x$. Then observe that

$$
\begin{aligned}
&\left| \langle e, e + P_r y - x \rangle - \langle f, f + P_r y - x \rangle \right| \\
=\ &\left| \langle f, e + P_r y - x \rangle - \langle f - e, e + P_r y - x \rangle - \langle f, f + P_r y - x \rangle \right| \\
=\ &\left| \langle f, e - f \rangle - \langle f - e, e + P_r y - x \rangle \right| \\
=\ &\left| \langle e + f + P_r y - x, e - f \rangle \right| \\
\leq\ &\left\| e + f + P_r y - x \right\| \cdot \left\| e - f \right\|.
\end{aligned}
\tag{3.5}
$$

Utilizing (3.1),(3.2) and the nonexpansivity of P_r we now compute

$$
\begin{aligned}
\left\| e + f + P_r y - x \right\| &= \left\| \sum_{i=1}^{r-1} \frac{a_i}{a_r} (P_i x - x) + x - P_r x + P_r y - x \right\| \\
&\leq \left\| \sum_{i=1}^{r-1} \frac{a_i}{a_r} (P_i x - x) \right\| + \left\| P_r y - P_r x \right\| \\
&\leq \sum_{i=1}^{r-1} \frac{a_i}{a_r} \left\| P_i x - x \right\| + \left\| P_r y - P_r x \right\| \\
&\leq \sum_{i=1}^{r-1} \frac{a_i}{a_r} \left\| P_i x - x \right\| + \left\| y - x \right\| \\
&\leq 4N (D + 1) + D \\
&\leq (D + 1)(4N + 1).
\end{aligned}
\tag{3.6}
$$

Substituting (3.6) in (3.5) and then utilizing (3.4), we get

$$
\left| \langle e, e + P_r y - x \rangle - \langle f, f + P_r y - x \rangle \right| \leq \frac{(D + 1)(4N + 1)\delta_{(\epsilon)}}{a_r}.
$$

This implies that

$$
\begin{aligned}
\langle x - P_r x, P_r y - P_r x \rangle \geq\ &\left\langle \sum_{i=1}^{r-1} \frac{a_i}{a_r} (P_i x - x), \sum_{i=1}^{r-1} \frac{a_i}{a_r} (P_i x - x) + P_r y - x \right\rangle \\
&- \frac{(D + 1)(4N + 1)\delta_{(\epsilon)}}{a_r}.
\end{aligned}
\tag{3.7}
$$

Note that

$$
\begin{aligned}
&\left\langle \sum_{i=1}^{r-1} \frac{a_i}{a_r} (P_i x - x), \sum_{i=1}^{r-1} \frac{a_i}{a_r} (P_i x - x) + P_r y - x \right\rangle \\
=\ &\left\langle \sum_{i=1}^{r-1} \frac{a_i}{a_r} (P_i x - x), \sum_{i=1}^{r-1} \frac{a_i}{a_r} (P_i x - x) \right\rangle + \left\langle \sum_{i=1}^{r-1} \frac{a_i}{a_r} (P_i x - x), P_r y - x \right\rangle \\
=\ &\left\| \sum_{i=1}^{r-1} \frac{a_i}{a_r} (P_i x - x) \right\|^2 + \sum_{i=1}^{r-1} \frac{a_i}{a_r} \left\| P_i x - x \right\|^2 + \sum_{i=1}^{r-1} \frac{a_i}{a_r} \langle P_i x - x, P_r y - P_i x \rangle.
\end{aligned}
$$

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Hence (3.7) now takes the form

$$\langle x - P_r x, P_r y - P_r x \rangle \geq \left\| \sum_{i=1}^{r-1} \frac{a_i}{a_r} (P_i x - x) \right\|^2 + \sum_{i=1}^{r-1} \frac{a_i}{a_r} \|P_i x - x\|^2$$

$$+ \sum_{i=1}^{r-1} \frac{a_i}{a_r} \langle P_i x - x, P_r y - P_i x \rangle - \frac{(D+1)(4N+1)\delta_{(\epsilon)}}{a_r} \quad (3.8)$$

$$\geq \left\| \sum_{i=1}^{r-1} \frac{a_i}{a_r} (P_i x - x) \right\|^2 + \sum_{i=1}^{r-1} \frac{a_i}{a_r} \|P_i x - x\|^2$$

$$+ \sum_{i=1}^{r-1} \frac{a_i}{a_r} \langle P_i x - x, P_i y - P_i x \rangle - \frac{5(D+1)(4N+1)\delta_{(\epsilon)}}{a_r} \quad (3.9)$$

since (using $\|P_r y - P_i y\| \leq \|P_r y - y\| + \|y - P_i y\| \leq 2\delta_{(\epsilon)}$)

$$\left| \sum_{i=1}^{r-1} \frac{a_i}{a_r} \langle P_i x - x, P_r y - P_i x \rangle - \sum_{i=1}^{r-1} \frac{a_i}{a_r} \langle P_i x - x, P_i y - P_i x \rangle \right| \leq 4N(D+1)2\delta_{(\epsilon)}$$

$$\leq \frac{16N(D+1)\delta_{(\epsilon)}}{a_r}.$$

Note that

$$\sum_{i=1}^{r-1} a_i \|P_i x - x\|^2 > 5(D+1)(4N+1)\delta_{(\epsilon)}$$

would lead to a contradiction of the conclusion in (3.9) since by the characterizing property of metric projections $\langle P_i x - x, P_i y - P_i x \rangle \geq 0$ while $\langle x - P_r x, P_r y - P_r x \rangle \leq 0$. Therefore

$$\|P_i x - x\|^2 \leq \frac{5(D+1)(4N+1)\delta_{(\epsilon)}}{a_i} \leq 5N(D+1)(4N+1)\delta_{(\epsilon)} \text{ for } 1 \leq i \leq r-1. \quad (3.10)$$

For

$$\delta_{(\epsilon)} \leq \frac{\epsilon^2}{5N(D+1)(4N+1)}, \quad (3.11)$$

we get

$$\|P_i x - x\| \leq \epsilon \text{ for } 1 \leq i \leq r-1.$$

For P_r (3.4) and (3.10) imply that

$$\|P_r x - x\| \leq \left\| \sum_{i=1}^{r-1} \frac{a_i}{a_r} (P_i x - x) \right\| + \frac{\delta_{(\epsilon)}}{a_r}$$

$$\leq \sum_{i=1}^{r-1} \frac{a_i}{a_r} \|P_i x - x\| + \frac{\delta_{(\epsilon)}}{a_r}$$

$$\leq 2N\sqrt{5N(D+1)(4N+1)\delta_{(\epsilon)}} + \frac{\delta_{(\epsilon)}}{a_r}.$$

Now in order to achieve that

$$2N\sqrt{5N(D+1)(4N+1)\delta_{(\epsilon)}} \leq \frac{39\epsilon}{40},$$

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it suffices that

$$\delta_{(\epsilon)} \leq \frac{\epsilon^2}{22N^3 \, (D+1) \, (4N+1)}. \tag{3.12}$$

Since such a $\delta_{(\epsilon)}$ trivially satisfies that $\delta_{(\epsilon)}/a_r \leq \epsilon/40$ we get

$$\|P_r x - x\| \leq \epsilon.$$

Observe that the choice of $\delta_{(\epsilon)}$ in (3.12) ultimately covers the choice in (3.11). This completes the proof of (i).

For (ii) we just have to observe that

$$\|Tx - x\| = \|\sum_{i=1}^{r} \alpha_i \lambda_i (P_i x - x)\|$$

and that the proof for (i), therefore, applies unchanged. $\qquad\square$

Remark 3.2.   (1) A bound D satisfying the condition in Theorem 3.1 can e.g. be always constructed when $C_{0,1}$ is bounded (and so in particular when all the C_i are bounded) since $D := diam(\{x\} \cup C_{0,1}) \geq diam(\{x\} \cup C_{0,\delta}) \geq dist(x, C_{0,\delta})$ for all $\delta \in (0,1]$ such that $C_{0,\delta} \neq \emptyset$.

(2) A kind of converse of Theorem 3.1 (even for nonexpansive retractions P_i) can easily be established as follows:

Let $P_i : H \to C_i$ be a nonexpansive retraction and let $x \in C_{0,\epsilon} \subseteq C_{i,\epsilon}$. Then there exists a $y \in C_i$ with $\|x - y\| < \epsilon$. This implies that x is a 2ϵ-common fixed point of P_i's since

$$
\begin{aligned}
\|P_i x - x\| & \leq & \|P_i x - P_i y\| + \|P_i y - y\| + \|y - x\| \\
& \leq & 2\|x - y\| \\
& < & 2\epsilon, \text{ for all } i = 1, 2, 3, \cdots, r.
\end{aligned}
$$

Since $0 \leq \lambda_i \leq 2$, therefore (1.3) now implies that

$$\|T_i x - x\| < 4\epsilon \text{ for all } i = 1, 2, 3, \cdots, r.$$

Hence, any $x \in C_{0,\delta_{(\epsilon)}}$ with $\delta_{(\epsilon)} := \frac{\epsilon}{4}$ is an ϵ-fixed point of the mapping T.

Recall that a normed linear space X is uniformly convex if for each ϵ with $0 < \epsilon \leq 2$ there corresponds a $\delta(\epsilon) > 0$ such that

$$\|x\|, \|y\| \leq 1 \text{ and } \|x - y\| \geq \epsilon \text{ implies that } \left\|\frac{x+y}{2}\right\| \leq 1 - \delta(\epsilon).$$

A mapping $\eta : (0,2] \to (0,1]$ which provides such a $\delta = \eta(\epsilon) > 0$ for a given $\epsilon \in (0,2]$ is known as a modulus of uniform convexity of X.

The next lemma is well-known:

Lemma 3.3. *Let $(X, \|\cdot\|)$ be a uniformly convex normed linear space with a modulus of uniform convexity η. If $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$ for a given $\epsilon \in (0,2]$, then*

$$\|\lambda x + (1 - \lambda)y\| \leq 1 - 2\lambda(1 - \lambda)\eta(\epsilon), \quad 0 \leq \lambda \leq 1.$$

In the spirit of Theorem 3.1, we now prove a uniform quantitative version of a result first established in [6] (see also [30] and – in the context of Busemann convex geodesic spaces – [24]). Here 'uniform' refers to the fact that the bound does not depend on $x \in C$. It is this fact which is crucially used in the next section and causes the need to upgrade

the assumption of strict convexity (made in the aforementioned references) to uniform convexity (see [19] for a detailed discussion of this point):

Theorem 3.4. *Let X be a uniformly convex normed linear space with modulus of uniform convexity η and let C be a nonempty convex subset of X. Let $T : C \to C$ be a mapping defined as $T := \sum_{i=1}^{r} \alpha_i T_i$, $0 < \alpha_i < 1$, $\sum_{i=1}^{r} \alpha_i = 1$ with $T_i := I + \lambda_i(P_i - I)$, $0 < \lambda_i < 1$ where $P_i : C \to C_i$ is a nonexpansive retraction of C onto some convex subset $C_i \subseteq C$. For every $\epsilon \in (0, 1]$ and $D \in \mathbb{N}$ there exists a $\theta_{(\epsilon)} > 0$, such that*

$$\forall x \in C \ \left(\|Tx - x\| \leq \theta_{(\epsilon)} \wedge C_{0,\theta_{(\epsilon)}} \cap C \neq \emptyset \wedge D > dist\left(x, C_{0,\theta_{(\epsilon)}} \cap C\right) \Longrightarrow x \in C_{0,\epsilon} \right),$$

where

$$\theta_{(\epsilon)} := \theta_{(\epsilon, N, D, \eta)} = \frac{2}{9N} \eta \left(\frac{\epsilon}{D + 1} \right) \epsilon,$$

and $N \in \mathbb{N}$

$$\frac{1}{N} \leq \min \left\{ \alpha_i \lambda_i (1 - \lambda_i) : 1 \leq i \leq r \right\}.$$

Proof. Let $\epsilon \in (0, 1]$. We aim to show that any $x \in C$ which is a $\theta_{(\epsilon)}$-fixed point of T is in fact an ϵ-common fixed point of P_i. As a consequence, $x \in C_{0,\epsilon}$. Now assume towards contradiction that $\|P_{i_0}x - x\| \geq \epsilon$ for some $i_0 \in \{1, 2, \cdots, r\}$. For the sake of notational simplicity, we assume that $i_0 = 1$. Let $y \in C_{0,\theta_{(\epsilon)}} \cap C$ such that $D \geq \|x - y\|$. Moreover, we may assume that $\|x - y\| > \frac{\epsilon}{3}$ for otherwise: $\|x - y\| \leq \frac{\epsilon}{3}$ implies that $x \in C_{0,\theta_{(\epsilon)}+\frac{\epsilon}{3}} \subseteq C_{0,\epsilon}$. Now observe that

$$\frac{\|P_1 x - x\|}{\|x - y\| + 2\theta_{(\epsilon)}} \geq \frac{\epsilon}{\|x - y\| + 2\theta_{(\epsilon)}} \geq \frac{\epsilon}{D + 1}. \tag{3.13}$$

Moreover,

$$\begin{aligned} \|P_1 x - y\| &\leq \|P_1 x - P_1 y\| + \|P_1 y - y\| \\ &\leq \|x - y\| + 2\theta_{(\epsilon)}. \end{aligned} \tag{3.14}$$

It follows from (3.13)-(3.14) and Lemma 3.3, that

$$\left\| (1 - \lambda_1) \frac{x - y}{\|x - y\| + 2\theta_{(\epsilon)}} + \lambda_1 \frac{P_1 x - y}{\|x - y\| + 2\theta_{(\epsilon)}} \right\| \leq 1 - 2\lambda_1 (1 - \lambda_1) \eta \left(\frac{\epsilon}{D + 1} \right). \tag{3.15}$$

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Utilizing (3.15), the following estimates hold:

$$
\begin{aligned}
\|x - y\| &\leq \|Tx - y\| + \theta_{(\epsilon)} \\
&= \left\| \sum_{i=1}^{r} \alpha_i T_i x - y \right\| + \theta_{(\epsilon)} \\
&= \left\| \sum_{i=1}^{r} \alpha_i \left((1 - \lambda_i)(x - y) + \lambda_i (P_i x - y) \right) \right\| + \theta_{(\epsilon)} \\
&\leq \sum_{i=1}^{r} \alpha_i \left\| (1 - \lambda_i)(x - y) + \lambda_i (P_i x - y) \right\| + \theta_{(\epsilon)} \\
&= \alpha_1 \left\| (1 - \lambda_1)(x - y) + \lambda_1 (P_1 x - y) \right\| \\
&\quad + \sum_{i=2}^{r} \alpha_i \left\| (1 - \lambda_i)(x - y) + \lambda_i (P_i x - P_i y + P_i y - y) \right\| + \theta_{(\epsilon)} \\
&\leq \alpha_1 \left(\|x - y\| + 2\theta_{(\epsilon)} - 2\lambda_1 (1 - \lambda_1) \eta \left(\frac{\epsilon}{D+1} \right) \left(\|x - y\| + 2\theta_{(\epsilon)} \right) \right) \\
&\quad + \sum_{i=2}^{r} \alpha_i \left\{ (1 - \lambda_i) \|x - y\| + \lambda_i \|P_i x - P_i y\| + 2\lambda_i \theta_{(\epsilon)} \right\} + \theta_{(\epsilon)} \\
&< \alpha_1 \left(\|x - y\| - \lambda_1 (1 - \lambda_1) \eta \left(\frac{\epsilon}{D+1} \right) \frac{2\epsilon}{3} \right) + \sum_{i=2}^{r} \alpha_i \|x - y\| + 3\theta_{(\epsilon)} \\
&= \|x - y\| - \alpha_1 \lambda_1 (1 - \lambda_1) \eta \left(\frac{\epsilon}{D+1} \right) \frac{2\epsilon}{3} + 3\theta_{(\epsilon)}.
\end{aligned}
\tag{3.16}
$$

Let $N \in \mathbb{N}$ be such that $\frac{1}{N} \leq \min \{ \alpha_i \lambda_i (1 - \lambda_i) : 1 \leq i \leq r \}$. Then for $\theta_{(\epsilon)} := \theta_{(\epsilon, N, D, \eta)} \leq \frac{2}{9N} \eta \left(\frac{\epsilon}{D+1} \right) \epsilon$, we have a contradiction. Therefore, any $x \in C$ which is a $\theta_{(\epsilon)}$-fixed point of T is an ϵ-common fixed point of P_i. Hence, $x \in C_{0,\epsilon}$. This completes the proof. $\square$

Remark 3.5. (i) Since $0 < \lambda_i < 1$, a converse of Theorem 3.4 easily follows as in Remark 3.2: any $x \in C_{0,\theta_{(\epsilon)}} \cap C$ with $\theta_{(\epsilon)} := \frac{\epsilon}{2}$ is an ϵ-fixed point of the mapping T.
(ii) If $\eta(\epsilon) = \epsilon \tilde{\eta}(\epsilon)$, where $\tilde{\eta}(\epsilon)$ is increasing as ϵ increases, we can improve our bound as follows: reasoning as in (3.16) (using that implies that (3.15) also holds with $\|x - y\| + 2\theta_{(\epsilon)}$ instead of $D + 1$) we get

$$
\begin{aligned}
\|x - y\| &< \alpha_1 \left(\|x - y\| - 2\lambda_1 (1 - \lambda_1) \eta \left(\frac{\epsilon}{\|x - y\| + 2\theta_{(\epsilon)}} \right) \left(\|x - y\| + 2\theta_{(\epsilon)} \right) \right) \\
&\quad + \sum_{i=2}^{r} \alpha_i \|x - y\| + 3\theta_{(\epsilon)} \\
&\leq \alpha_1 \left(\|x - y\| - 2\lambda_1 (1 - \lambda_1) \tilde{\eta} \left(\frac{\epsilon}{\|x - y\| + 2\theta_{(\epsilon)}} \right) \epsilon \right) + \sum_{i=2}^{r} \alpha_i \|x - y\| + 3\theta_{(\epsilon)} \\
&\leq \alpha_1 \left(\|x - y\| - 2\lambda_1 (1 - \lambda_1) \tilde{\eta} \left(\frac{\epsilon}{D+1} \right) \epsilon \right) + \sum_{i=2}^{r} \alpha_i \|x - y\| + 3\theta_{(\epsilon)}.
\end{aligned}
$$

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Again, letting $\frac{1}{N} \leq \min \{\alpha_i \lambda_i (1 - \lambda_i) : 1 \leq i \leq r\}$ and $\theta_{(\epsilon)} := \theta_{(\epsilon,N,D,\tilde{\eta})} \leq \frac{2}{3N} \tilde{\eta} \left(\frac{\epsilon}{D+1} \right) \epsilon$, we have a contradiction.

4. Applications

So far we have shown that
(i) the iterates of the mapping T defined in (1.2) and (1.4) are asymptotically regular with the bounds $\Phi(2d, \epsilon, N)$ and $\Phi\left(2d, \epsilon, (2N+1)^2\right)$ resp. $\Phi\left(2d, \epsilon, (N+1)^2\right)$;
(ii) the set of approximate fixed points of the respective mappings correspond uniformly to the points in $C_{0,\epsilon}$ in the setting of Hilbert spaces and uniformly convex normed linear spaces.

This collectively implies the following result:

Theorem 4.1. *(i)* Let H be a Hilbert space and let $T : H \to H$ be a mapping defined as $T := \sum_{i=1}^{r} \alpha_i T_i$, $0 < \alpha_i < 1, \sum_{i=1}^{r} \alpha_i = 1$ with $T_i := I + \lambda_i(P_i - I)$, $0 < \lambda_i \leq 2, \lambda_1 < 2$, where $P_i : H \to C_i$ is a metric projection of H onto some closed convex subset $C_i \subseteq H$ and $F(T) \neq \emptyset$. Let $x_n = T^n x_0$ for some $x_0 \in H$ and $\epsilon \in (0, 1]$, then

$$\forall n \geq \Psi(d, D, N_1, N_2, \epsilon) \ (x_n \in C_{0,\epsilon})$$

with

$$\Psi(d, D, N_1, N_2, \epsilon) := \left\lceil \frac{1936 \cdot d^2 \cdot N_1^6 (2d + D + 1)^2 (4N_1 + 1)^2 \cdot (2N_2 + 1)^2}{\pi \cdot \epsilon^4} \right\rceil,$$

where $d > \|x_0 - p\|$ for some $p \in F(T)$, $D > dist\left(x_0, C_{0,\delta_{(\epsilon,N_1,2d+D)}}\right)$, $C_{0,\delta_{(\epsilon,N_1,2d+D)}} \neq \emptyset$, $N_1, N_2 \in \mathbb{N}$ such that $\frac{1}{N_1} \leq \min \{\alpha_i \lambda_i : 1 \leq i \leq r\}$, $\frac{1}{N_2} \leq \min \{\alpha_1, 2 - \lambda_1\}$ and δ as in Theorem 3.1.
(ii) '(i)' also holds for T defined as $T := \alpha_0 I + \sum_{i=1}^{r} \alpha_i T_i$ with T_i as before and $\alpha_0, \ldots, \alpha_r \in (0, 1)$ with $\sum_{i=0}^{r} \alpha_i = 1$, where now

$$\Psi(d, D, N_1, N_2, \epsilon) := \left\lceil \frac{1936 \cdot d^2 \cdot N_1^6 (2d + D + 1)^2 (4N_1 + 1)^2 \cdot N_2}{\pi \cdot \epsilon^4} \right\rceil,$$

with $N_1, N_2 \in \mathbb{N}$ such that $\frac{1}{N_1} \leq \min \{\alpha_i \lambda_i : 1 \leq i \leq r\}$ and $\frac{1}{N_2} \leq \alpha_0$.
(iii) Let X be a uniformly convex normed linear space with modulus of uniform convexity η and let C be a nonempty convex subset of X. Let $T : C \to C$ be a mapping defined as $T := \sum_{i=1}^{r} \alpha_i T_i$, $0 < \alpha_i < 1, \sum_{i=1}^{r} \alpha_i = 1$ with $T_i := I + \lambda_i(P_i - I), 0 < \lambda_i < 1$ where $P_i : C \to C_i$ is a nonexpansive retraction of C onto some convex subset $C_i \subseteq C$ and $F(T) \neq \emptyset$. Let $x_n = T^n x_0$ for some $x_0 \in C$ and $\epsilon \in (0, 1]$, then

$$\forall n \geq \Psi(d, D, N_1, N_2, \eta, \epsilon) \ (x_n \in C_{0,\epsilon})$$

with

$$\Psi(d, D, N_1, N_2, \eta, \epsilon) := \left\lceil \frac{81 \cdot d^2 \cdot N_1^2 \cdot (N_2 + 1)^2}{\pi \cdot \left(\eta(\frac{\epsilon}{2d+D+1}) \right)^2 \cdot \epsilon^2} \right\rceil,$$

where $d > \|x_0 - p\|$ for some $p \in F(T)$, $D > dist\left(x_0, C_{0,\theta_{(\epsilon,N_1,2d+D,\eta)}} \cap C\right)$, $C_{0,\theta_{(\epsilon,N_1,2d+D,\eta)}} \cap C \neq \emptyset$, $N_1, N_2 \in \mathbb{N}$ such that $\frac{1}{N_1} \leq \min \{\alpha_i \lambda_i (1 - \lambda_i) : 1 \leq i \leq r\}$, $\frac{1}{N_2} \leq \min\{\alpha_1, 1 - \lambda_1\}$ and θ as in Theorem 3.4.

Proof. The asymptotic regularity of the iterates of the mapping T defined in (i)-(iii) follows from Theorem 2.3 (for (i)), Theorem 2.1 (for (ii)) and Theorem 2.5 (for (iii)) with the bounds $\Phi\left(2d, \epsilon, (2N_2+1)^2\right)$, $\Phi\left(2d, \epsilon, N_2\right)$, resp. $\Phi\left(2d, \epsilon, (N_2+1)^2\right)$.

For the desired results in (i),(ii) utilize Theorem 3.1(i),(ii) by letting $\epsilon := \delta_{(\epsilon, N_1, 2d+D)}$ since $\|x_n - x_0\| \leq 2d$ implies that

$$dist(x_n, C_{0,\delta_{(\epsilon)}}) \leq 2d + dist(x_0, C_{0,\delta_{(\epsilon)}}) < 2d + D.$$

The result in (iii) follows from utilizing Theorem 3.4 with $\epsilon := \theta_{(\epsilon, N_1, D, \eta)}$.
An easy calculation shows that

$$\Psi(d, D, N_1, N_2, \epsilon) = \Phi(2d, \delta_{(\epsilon, N_1, 2d+D)}, (2N_2+1)^2) \qquad \text{in (i),} \qquad (4.1)$$

$$\Psi(d, D, N_1, N_2, \epsilon) = \Phi(2d, \delta_{(\epsilon, N_1, 2d+D)}, N_2) \qquad \text{in (ii), and} \qquad (4.2)$$

$$\Psi(d, D, N_1, N_2, \eta, \epsilon) = \Phi(2d, \theta_{(\epsilon, N_1, 2d+D, \eta)}, (N_2+1)^2) \quad \text{(in (iii)).} \qquad (4.3)$$

$\square$

Remark 4.2. Theorem 4.1 provides a solution to the problem of image recovery up to an ϵ-perturbation of the original problem even in cases where the original problem has no solution because of $C_0 = \emptyset$ since the condition $F(T) \neq \emptyset \wedge C_{0,\delta} \neq \emptyset$ is weaker than
(i) $C_0 = F(T) = \cap_{i=1}^r F(T_i) \neq \emptyset$ used in Theorem 1.1 and
(ii) $C_0 = \cap_{i=1}^r F(T_i) \neq \emptyset$ used in Theorem 1.2.

The following example supports the above assertion:
Let $X := \mathbb{R}$, $C_1 := \{c\}, C_2 := \{-c\}$ for some $c \neq 0$ and let P_i be the metric projection of $\mathbb{R}$ onto the closed convex set C_i. For $i = 1, 2$, let $T_i : \mathbb{R} \to \mathbb{R}$ be defined by

$$T_i = \frac{1}{2}I + \frac{1}{2}P_i,$$

with

$$T = \frac{1}{2}T_1 + \frac{1}{2}T_2.$$

Then $F(T_1) = \{c\}$ and $F(T_2) = \{-c\}$ so that $F(T_1) \cap F(T_2) = C_0 = \emptyset$. However, $F(T) = \{0\}$. Moreover, for $|c| < \delta$ then $0 \in C_{0,\delta} \neq \emptyset$.

We now present a weaker version of Theorem 4.1 when $C_0 \neq \emptyset$ since it is easier to state:

Corollary 4.3. *(i) Let H be a Hilbert space and let $T : H \to H$ be a mapping defined as $T := \sum_{i=1}^r \alpha_i T_i$, $0 < \alpha_i < 1, \sum_{i=1}^r \alpha_i = 1$ with $T_i := I + \lambda_i(P_i - I)$, $0 < \lambda_i \leq 2, \lambda_1 < 2$, where $P_i : H \to C_i$ is a metric projection of H onto some closed convex subset $C_i \subseteq H$ and $F(T) \neq \emptyset$. Let $x_n = T^n x_0$ for some $x_0 \in H$ and $\epsilon \in (0, 1]$, then*

$$\forall n \geq \Psi(D, N_1, N_2, \epsilon) \ (x_n \in C_{0,\epsilon}),$$

with

$$\Psi(d, D, N_1, N_2, \epsilon) := \left\lceil \frac{1936 \cdot N_1^6 \cdot (D+1)^4 (4N_1+1)^2 \cdot (2N_2+1)^2}{\pi \cdot \epsilon^4} \right\rceil,$$

where $D > \|x_0 - p\|$ for some $p \in C_0$; $N_1, N_2 \in \mathbb{N}$ such that $\frac{1}{N_1} \leq \min\{\alpha_i \lambda_i : 1 \leq i \leq r\}$ and $\frac{1}{N_2} \leq \min\{\alpha_1, 2 - \lambda_1\}$.

(ii) '(i)' also holds for T defined as $T := \alpha_0 I + \sum_{i=1}^{r} \alpha_i T_i$ with T_i as before and $\alpha_0, \ldots, \alpha_r \in (0,1)$ with $\sum_{i=0}^{r} \alpha_i = 1$, where now

$$\Psi(d, D, N_1, N_2, \epsilon) := \left\lceil \frac{1936 \cdot N_1^6 (D+1)^4 (4N_1+1)^2 \cdot N_2}{\pi \cdot \epsilon^4} \right\rceil,$$

with $N_1, N_2 \in \mathbb{N}$ such that $\frac{1}{N_1} \leq \min\{\alpha_i \lambda_i : 1 \leq i \leq r\}$ and $\frac{1}{N_2} \leq \alpha_0$.

(iii) Let X be a uniformly convex normed linear space with modulus of uniform convexity η and let C be a nonempty convex subset of X. Let $T : C \to C$ be a mapping defined as $T := \sum_{i=1}^{r} \alpha_i T_i$, $0 < \alpha_i < 1, \sum_{i=1}^{r} \alpha_i = 1$ with $T_i := I + \lambda_i (P_i - I), 0 < \lambda_i < 1$ where $P_i : C \to C_i$ is a nonexpansive retraction of C onto some convex subset $C_i \subseteq C$ and $C_0 := \cap_{i=1}^{r} F(T_i) \neq \emptyset$. Let $x_n = T^n x_0$ for some $x_0 \in C$ and $\epsilon \in (0,1]$, then

$$\forall n \geq \Psi(D, N_1, N_2, \eta, \epsilon) \ (x_n \in C_{0,\epsilon})$$

with

$$\Psi(D, N_1, N_2, \eta, \epsilon) := \left\lceil \frac{81 \cdot D^2 \cdot N_1^2 \cdot (N_2+1)^2}{\pi \cdot \left(\eta(\frac{\epsilon}{D+1})\right)^2 \cdot \epsilon^2} \right\rceil,$$

where $D > \|x_0 - p\|$ for some $p \in C_0$; $N_1, N_2 \in \mathbb{N}$ such that $\frac{1}{N_1} \leq \min\{\alpha_i \lambda_i (1 - \lambda_i) : 1 \leq i \leq r\}$ and $\frac{1}{N_2} \leq \min\{\alpha_1, 1 - \lambda_1\}$.

Proof. The proof follows from the previous one, since we now may take $p \in C_0 \subseteq F(T)$ and hence may use D as upper bound for $\|x_0 - p\|$ and - using that $C_0 \subseteq C_{0,\delta}(\cap C)$ (for any $\delta > 0$) - we get $D > \|x_0 - p\| \geq \|x_n - p\| \geq dist(x_n, C_{0,\delta}(\cap C))$ so that we can replace '$2d + D$' by 'D' in the previous proof. $\qquad\square$

Remark 4.4. Note that L^p-spaces $(1 < p < \infty)$ are uniformly convex with modulus of uniform convexity $\eta(\epsilon) := \frac{1}{p}\left(\frac{\epsilon}{2}\right)^p$ for $p \geq 2$. As $\eta(\epsilon) := \epsilon \tilde{\eta}(\epsilon)$, therefore we get $\tilde{\eta}(\epsilon) := \frac{1}{p}\left(\frac{\epsilon^{p-1}}{2^p}\right)$. Now $\theta_{(\epsilon, N, D, \eta)}$ from Remark 3.5(ii) simplifies to

$$\theta_{(\epsilon, N, D)} := \frac{2}{3N \cdot p \cdot (D+1)^{p-1}} \left(\frac{\epsilon}{2}\right)^p.$$

As a consequence, we get the bound $\Psi(D, N_1, N_2, \epsilon) := \frac{9 \cdot 2^{2p} \cdot p^2 \cdot (D+1)^{2p} \cdot N_1^2 \cdot (N_2+1)^2}{\pi \cdot \epsilon^{2p}}$ in (iii) for L^p-spaces. For Hilbert space this gives a bound of order ϵ^4 as in (i) but only for $\lambda_i \in (0,1)$.

Typos (also in published version, 9.6.2014): In Corollary 4.3(i) replace '$F(T) \neq \emptyset$' by '$C_0 \neq \emptyset$' and in Corollary 4.3(i),(ii) '$\Psi(d, D, N_1, N_2, \epsilon) :=$' by '$\Psi(D, N_1, N_2, \epsilon) :=$'.

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