ON MODIFIED HALPERN AND TIKHONOV-MANN ITERATIONS

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Abstract. We show that the asymptotic regularity and the strong convergence of the modified Halpern iteration due to T.-H. Kim and H.-K. Xu and studied further by A. Cuntavenapit and B. Panyanak and the Tikhonov-Mann iteration introduced by H. Cheval and L. Leuştean as a generalization of an iteration due to Y. Yao et al. that has recently been studied by Boţ et al. can be reduced to each other in general geodesic settings. This, in particular, gives a new proof of the convergence result in Boţ et al. together with a generalization from Hilbert to CAT(0) spaces. Moreover, quantitative rates of asymptotic regularity and metastability due to K. Schade and U. Kohlenbach can be adapted and transformed into rates for the Tikhonov-Mann iteration corresponding to recent quantitative results on the latter of H. Cheval, L. Leuştean and B. Dinis, P. Pinto respectively. A transformation in the converse direction is also possible. We also obtain rates of asymptotic regularity of order $O(1/n)$ for both the modified Halpern (and so in particular for the Halpern iteration) and the Tikhonov-Mann iteration in a general geodesic setting for a special choice of scalars.

Keywords: Mann iteration; Halpern iteration; Tikhonov regularization; Rates of asymptotic regularity; Rates of metastability; Proof mining.

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1. Introduction

We consider in the sequel generalizations of the well-known Mann and Halpern iterations obtained by combining them with the so-called Tikhonov regularization terms [3, 23]. Although we will in the rest of the paper work in a general geodesic setting we first discuss these iterations for simplicity in the context of linear normed spaces, where X is a Banach space, $C \subseteq X$ is a convex subset, and $T : C \to C$ is a nonexpansive mapping.

One such generalization is the *Tikhonov-Mann iteration*, defined in [7] as follows:

$$(1) \qquad x_{n+1} = (1-\lambda_n)((1-\beta_n)u + \beta_n x_n) + \lambda_n T\big((1-\beta_n)u + \beta_n x_n\big),$$

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where $(\lambda_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}$ are sequences in $[0,1]$ and $x_0, u \in C$. Obviously, if $\beta_n = 1$, then (x_n) becomes the Mann iteration. For $u = 0$ one gets a modified Mann iteration, studied in [32] and rediscovered in a recent paper [4].
Another generalization is the *modified Halpern iteration*, introduced in [13]:

$$y_{n+1} := \gamma_n v + (1-\gamma_n)(\alpha_n y_n + (1-\alpha_n)Ty_n), \tag{2}$$

where $(\gamma_n)_{n\in\mathbb{N}}$ and $(\alpha_n)_{n\in\mathbb{N}}$ are sequences in $[0,1]$ and $v, y_0 \in C$.
[13] showed the strong convergence of (y_n) in uniformly smooth Banach spaces under certain conditions on the scalars and assuming that T has a fixed point. Under somewhat more liberal conditions, [9] showed the strong convergence in the nonlinear setting of CAT(0) spaces.

In this paper we establish, in a general nonlinear setting, a strong connection between the modified Halpern and the Tikhonov-Mann iteration schemes. From this connection it follows that the strong convergence of one scheme implies that of the other. In particular, the strong convergence theorem of [4] follows from the much older results on the modified Halpern iteration and - by a slight modification of the argument provided in [9] - also under the exact same conditions on the scalars as assumed in [4]. Moreover, quantitative rates of asymptotic regularity and metastability (in the sense of Tao [30, 31]) for one scheme translate into corresponding rates for the other scheme. In 2012, Schade and the second author [28] extracted rates of asymptotic regularity for the modified Halpern iteration in a general nonlinear setting and of metastability for CAT(0) spaces from the proof given in [9]. We show that with a slight modification of the rate of asymptotic regularity one can in that extraction weaken again the conditions on the scalars to those used in [4]. By the aforementioned reduction of the Tikonov-Mann iteration to the modified Halpern iteration this induces corresponding rates of asymptotic regularity in a general nonlinear setting and of metastability for CAT(0) spaces for the Tikhonov-Mann scheme. A rate of asymptotic regularity in this case has recently been extracted in [7] directly from the proof given in [4] by the first and the third author and a rate of metastability has recently been obtained for the modified Mann iteration, a special case of the Tikhonov-Mann iteration, in the case of Hilbert spaces in [10]. In [11] the authors introduce an alternating Halpern-Mann iteration and compute rates of metastability for this iteration in the setting of CAT(0) spaces. As the Tikhonov-Mann iteration (x_n) is a special case of the alternating Halpern-Mann iteration, one gets, as a corollary of [11, Theorem 5.1] rates of metastability for (x_n). However, the proof of [11, Theorem 5.1] uses a stronger condition on the scalars than in our result.
Conversely, rates for the Tikhonov-Mann iteration imply - by the connection established in this paper - corresponding rates for the modified Halpern iteration.
For some special test case for the choice of scalars, we for the first time obtain rates of asymptotic regularity of order $O(1/n)$ for both iterations which are new even in the linear case.

2. W-spaces

Firstly, let us recall some basic notions from geodesic geometry. We refer to [26] for details. Let (X,d) be a metric space. A geodesic path (or simply a geodesic) in X is a function $\gamma : [a,b] \to X$ which is distance-preserving, that is $d(\gamma(s),\gamma(t)) = |s-t|$ for all $s,t \in [a,b]$. A geodesic segment in X is the image of a geodesic in X.

If $\gamma : [a, b] \to X$ is a geodesic, $\gamma(a) = x$ and $\gamma(b) = y$, we say that the geodesic γ joins x and y or that the geodesic segment $\gamma([a, b])$ joins x and y. The metric space (X, d) is (uniquely) geodesic if every two points of X are joined by a (unique) geodesic segment. The following useful properties are well-known.

Lemma 2.1. *Assume that X is a geodesic space.*

(i) *Let $x, y \in X$ and $\gamma([a, b])$ be a geodesic segment that joins x and y. For every $\lambda \in [0, 1]$, $z = \gamma((1-\lambda)a + \lambda b)$ is the unique point in $\gamma([a, b])$ satisfying $d(z, x) = \lambda d(x, y)$, and this unique z satisfies also $d(z, y) = (1-\lambda)d(x, y)$.*

(ii) *The following are equivalent:*
   (a) *X is uniquely geodesic.*
   (b) *For any $x, y \in X$ and any $\lambda \in [0, 1]$ there exists a unique element $z \in X$ such that $d(x, z) = \lambda d(x, y)$ and $d(y, z) = (1-\lambda)d(x, y)$.*

As in [7], we consider a W-space to be a metric space (X, d) together with a function $W : X \times X \times [0, 1] \to X$. We think of $W(x, y, \lambda)$ as an abstract convex combination of the points $x, y \in X$. That is why we shall write $(1-\lambda)x+\lambda y$ instead of $W(x, y, \lambda)$. In the sequel, we denote a W-space simply by X. Let us define, for any $x, y \in X$, $[x, y] = \{(1-\lambda)x + \lambda y \mid \lambda \in [0, 1]\}$. A nonempty subset $C \subseteq X$ is said to be convex if for all $x, y \in C$, we have that $[x, y] \subseteq C$.
Consider the following axioms:

(W1) $d(z, (1-\lambda)x + \lambda y) \le (1-\lambda)d(z, x) + \lambda d(z, y)$,

(W2) $d((1-\lambda)x + \lambda y, (1-\tilde{\lambda})x + \tilde{\lambda}y) = |\lambda - \tilde{\lambda}| d(x, y)$,

(W3) $(1-\lambda)x + \lambda y = \lambda y + (1-\lambda)x$,

(W4) $d((1-\lambda)x + \lambda z, (1-\lambda)y + \lambda w) \le (1-\lambda)d(x, y) + \lambda d(z, w)$,

(W5) $1x + 0y = x$ and $0x + 1y = y$,

(W6) $(1-\lambda)x + \lambda x = x$,

(W7) $d(x, (1-\lambda)x + \lambda y) = \lambda d(x, y)$ and $d(y, (1-\lambda)x + \lambda y) = (1-\lambda)d(x, y)$.

W-spaces satisfying (W1) were introduced by Takahashi [29] under the name of convex metric spaces.

2.1. W-geodesic spaces.

Definition 2.2. *A W-geodesic space is a W-space X satisfying* $(W2)$ *and* $(W5)$.

Let X be a W-geodesic space.

Proposition 2.3. *(W6) and (W7) hold.*

Proof. Use that, by (W5), $x = 1x+0x = 1x+0y$, $y = 0x+1y$, and apply (W2). ☐

Define, for any $x \ne y \in X$, the mapping

$$W_{xy} : [0, d(x, y)] \to X, \qquad W_{xy}(s) = \left(1 - \frac{s}{d(x, y)}\right)x + \frac{s}{d(x, y)}y.$$

We also define, for uniformity, $W_{xx} : \{0\} \to X,\ W_{xx}(0) = x$.

Proposition 2.4. *For all $x, y \in X$, W_{xy} is a geodesic that joins x and y such that $W_{xy}([0, d(x, y)]) = [x, y]$. Thus, $[x, y]$ is a geodesic segment that joins x and y.*

Proof. The case $x = y$ is trivial, by (W6). Assume that $x \neq y$. One can easily see that, by (W2), W_{xy} is a geodesic. Furthermore, by (W5), we have that $W_{xy}(0) = x$ and $W_{xy}(d(x,y)) = y$. Thus, W_{xy} is a geodesic that joins x and y. Since the mapping $[0,1] \to [0, d(x,y)]$, $\lambda \mapsto \lambda d(x,y)$ is a bijection, it follows that $W_{xy}([0, d(x,y)]) = [x,y]$. □

In fact, a metric space is geodesic if and only if it is W-geodesic for some $W : X \times X \times [0,1] \to X$.

Proposition 2.5. *For all $x, y \in X$ and all $\lambda \in [0,1]$, there exists a unique $z \in [x,y]$ (namely $z = (1-\lambda)x + \lambda y$) such that*

$$d(x,z) = \lambda d(x,y) \text{ and } d(y,z) = (1-\lambda)d(x,y). \tag{3}$$

Proof. Apply (W7), Lemma 2.1.(i) and the previous proposition. □

2.2. **W-hyperbolic spaces.** A W-hyperbolic space [14] is a W-space satisfying (W1)-(W4). One can easily see that (W5)-(W7) also hold in a W-hyperbolic space. In particular, any W-hyperbolic space is a W-geodesic space.
W-hyperbolic spaces turn out to be a natural class of geodesic spaces for the study of nonlinear iterations. Normed spaces are obvious examples of W-hyperbolic spaces, as one can define $W(x,y,\lambda) = (1-\lambda)x + \lambda y$. Busemann spaces [6, 26] and CAT(0) spaces [1, 5] are also W-hyperbolic spaces:

(i) by [2, Proposition 2.6], Busemann spaces are the uniquely geodesic W-hyperbolic spaces;
(ii) by [15, p. 386-388], CAT(0) spaces are the W-hyperbolic spaces X satisfying

$$d^2\left(z, \frac{1}{2}x + \frac{1}{2}y\right) \le \frac{1}{2}d^2(z,x) + \frac{1}{2}d^2(z,y) - \frac{1}{4}d^2(x,y) \quad \text{for all } x,y,z \in X.$$

It is well-known that any CAT(0) space is a Busemann space.

3. The Tikhonov-Mann and modified Halpern iterations

Let X be a W-space, $C \subseteq X$ a convex subset, and $T : C \to C$ be a nonexpansive mapping, i.e. for all $x, y \in C$

$$d(Tx, Ty) \le d(x,y).$$

This very general setting suffices for defining the iterations of interest for us in this paper. Let (β_n) and (λ_n) be sequences in $[0,1]$ and $u \in C$.
The *Tikhonov-Mann iteration* (x_n) and the *modified Halpern iteration* (y_n) are defined as follows:

$$x_0 \in C, \qquad x_{n+1} = (1-\lambda_n)u_n + \lambda_n T u_n, \tag{4}$$

$$y_0 \in C, \qquad y_{n+1} = (1-\beta_{n+1})u + \beta_{n+1} v_n, \tag{5}$$

where

$$u_n = (1-\beta_n)u + \beta_n x_n \quad \text{and} \quad v_n = (1-\lambda_n)y_n + \lambda_n T y_n. \tag{6}$$

The ordinary Halpern iteration is the special case of (y_n) with $\lambda_n = 1$ for all $n \in \mathbb{N}$ and was introduced (in the case where $u := 0$) by Halpern in [12].

Remark 3.1. *We use for the parameter sequences from the definition of the modified Halpern iteration different notations than the ones from* [13, 9, 28]*: we write* $1-\beta_{n+1}$ *instead of* β_n *and* $1-\lambda_n$ *instead of* α_n.

We get from (4) the following inductive definition for (u_n):

$$\begin{aligned} u_0 &= (1-\beta_0)u + \beta_0 x_0 \in C, \\ u_{n+1} &= (1-\beta_{n+1})u + \beta_{n+1}\big((1-\lambda_n)u_n + \lambda_n T u_n\big). \end{aligned} \tag{7}$$

The following observation establishes the essential link between our iterations.

Proposition 3.2. *Assume that* $y_0 = (1-\beta_0)u + \beta_0 x_0$. *Then for all* $n \in \mathbb{N}$,

$$u_n = y_n \text{ and } x_{n+1} = v_n.$$

Proof. We prove the first equality by induction on n. The case $n = 0$ holds by hypothesis. As for the inductive step $n \Rightarrow n+1$:

$$\begin{aligned} y_{n+1} &\overset{(5)}{=} (1-\beta_{n+1})u + \beta_{n+1}v_n \overset{(6)}{=} (1-\beta_{n+1})u + \beta_{n+1}\big((1-\lambda_n)y_n + \lambda_n T y_n\big) \\ &\overset{\text{(IH)}}{=} (1-\beta_{n+1})u + \beta_{n+1}\big((1-\lambda_n)u_n + \lambda_n T u_n\big) \\ &\overset{(7)}{=} u_{n+1}. \end{aligned}$$

Furthermore, $x_{n+1} = (1-\lambda_n)u_n + \lambda_n T u_n = (1-\lambda_n)y_n + \lambda_n T y_n = v_n$. □

3.1. **(Quantitative) conditions on (β_n) and (λ_n).** The following conditions on the sequences (β_n) and (λ_n) were used in the study of the asymptotic regularity and strong convergence of the Tikhonov-Mann and modified Halpern iterations:

$$\begin{array}{llll} (H1) & \displaystyle\sum_{n=0}^{\infty}(1-\beta_n) = \infty, & (H1^*) & \displaystyle\prod_{n=1}^{\infty}\beta_{n+1} = 0, \\ (H2) & \displaystyle\sum_{n=0}^{\infty}|\beta_{n+1}-\beta_n| < \infty, & (H3) & \displaystyle\sum_{n=0}^{\infty}|\lambda_{n+1}-\lambda_n| < \infty, \\ (H4) & \displaystyle\lim_{n\to\infty}\beta_n = 1, & (H5) & \displaystyle\liminf_{n\to\infty}\lambda_n > 0, \\ (H6) & \displaystyle\lim_{n\to\infty}\lambda_n = 1, & (H7) & \displaystyle\sum_{n=0}^{\infty}(1-\lambda_n) = \infty. \end{array}$$

Kim and Xu [13] proved the strong convergence of the modified Halpern iteration in uniformly smooth Banach spaces under the hypotheses $(H1)-(H4)$, $(H6)$, and $(H7)$. Cuntavenapit and Panyanak [9] showed that strong convergence holds in CAT(0) spaces without assuming $(H7)$; they remarked that $(H7)$ can be eliminated also in the case of Kim and Xu's result. Note that all the remaining conditions permit the choice of $\lambda_n = 1$ (for all n) by which the modified Halpern iteration becomes the ordinary Halpern iteration.

It is easy to see that the proof in [9] can be modified in such a way that instead of $(H6)$ only the weaker condition $(H5)$ is needed: replace the last inequality in [9, (3.4)] by (using $(H5)$ for $\lambda_n = 1-\alpha_n$)

$$(1-\alpha_n)d(x_n, Tx_n) \le d(x_n, x_{n+1}) + \beta_n d(u, y_n) \to 0, \text{ as } n \to \infty.$$

Boţ, Csetnek and Meier [4] used $(H1)-(H5)$ to obtain the strong convergence of a modified Mann iteration in Hilbert spaces. By the comment above, Proposition 3.2 and Lemma 4.1 below, this result follows from [9].

As a consequence of the quantitative results obtained by Cheval and Leuştean [7], conditions $(H1)-(H5)$ suffice for proving the asymptotic regularity of the Tikhonov-Mann iteration in W-hyperbolic spaces. We also consider condition $(H1^*)$ (which - for strictly positive $\beta_n > 0$ - is equivalent with $(H1)$), since, as observed for the first time by the second author [16], its quantitative version is useful in obtaining better rates of asymptotic regularity.
As we are interested in effective bounds on the asymptotic behaviour of our iterations, we consider quantitative versions of the above conditions (with the exception of $(H7)$, which, as pointed above, is superfluous):

$(H1_q)$ $\sum_{n=2}^{\infty}(1-\beta_n)$ diverges with rate of divergence σ_1;

$(H1_q^*)$ $\prod_{n=1}^{\infty}\beta_{n+1}=0$ with rate of convergence σ_1^*;

$(H2_q)$ $\sum_{n=0}^{\infty}|\beta_{n+1}-\beta_n|$ converges with Cauchy modulus σ_2;

$(H3_q)$ $\sum_{n=0}^{\infty}|\lambda_{n+1}-\lambda_n|$ converges with Cauchy modulus σ_3;

$(H4_q)$ $\lim_{n\to\infty}\beta_n=1$ with rate of convergence σ_4;

$(H5_q)$ $\Lambda\in\mathbb{N}^*$ and $N_\Lambda\in\mathbb{N}$ are such that $\lambda_n\geq\frac{1}{\Lambda}$ for all $n\geq N_\Lambda$;

$(H6_q)$ $\lim_{n\to\infty}\lambda_n=1$ with rate of convergence σ_5.

We refer, for example, to [7] for the definitions of quantitative notions such as rate of convergence, Cauchy modulus, rate of divergence. The indices in our conditions above are chosen in such a way that the respective moduli satisfy the conditions in both [7] and [28].

4. Rates of asymptotic regularity

Let us recall that if X is a metric space, $\emptyset\neq C\subseteq X$, and $T:C\to C$, then a sequence (a_n) in C is said to be

(i) asymptotically regular if $\lim_{n\to\infty}d(a_n,a_{n+1})=0$; a rate of asymptotic regularity of (a_n) is a rate of convergence of $(d(a_n,a_{n+1}))$ towards 0.
(ii) T-asymptotically regular if $\lim_{n\to\infty}d(a_n,Ta_n)=0$; a rate of T-asymptotic regularity of (a_n) is a rate of convergence of $(d(a_n,Ta_n))$ towards 0.

In the sequel, we explore the relation between rates of (T-)asymptotic regularity of the Tikhonov-Mann iteration (x_n) and those of the modified Halpern iteration (y_n).
For the rest of the section, (X,d,W) is a W-hyperbolic space, C is a convex subset of X, and $T:C\to C$ is a nonexpansive mapping. We assume that T has fixed points, hence the set $Fix(T)$ of fixed points of T is nonempty. If p is a fixed point of T, define

$$M_p=\max\{d(x_0,p),d(u,p)\}. \tag{8}$$

By [7, Lemma 3.1.(ii)] and [7, Proposition 3.2.(8)], we have that

$$d(x_n,u_n)\leq 2M_p(1-\beta_n). \tag{9}$$

Let $K\in\mathbb{N}^*$ be such that $K\geq M_p$.

Lemma 4.1. *Assume that $(H4_q)$ holds. Then $\lim_{n\to\infty} d(x_n,u_n) = 0$ with rate of convergence*

$$\alpha(k) = \sigma_4\left(2K(k+1)-1\right). \tag{10}$$

Proof. Let $n \geq \alpha(k)$. Then, by (9), we get that

$$d(x_n,u_n) \leq 2M_p(1-\beta_n) \leq 2K(1-\beta_n) \leq \frac{2K}{2K(k+1)} = \frac{1}{k+1}.$$

□

Proposition 4.2. *Assume that $(H4_q)$ holds and let $\Phi:\mathbb{N}\to\mathbb{N}$. Define $\Phi':\mathbb{N}\to\mathbb{N}$ by*

$$\Phi'(k) := \max\left\{\alpha(3k+2), \Phi(3k+2)\right\}, \tag{11}$$

where α is given by (10).

(i) *If Φ is a rate of (T-)asymptotic regularity of one of the sequences (x_n), (u_n), then Φ' is a rate of (T-)asymptotic regularity of the other one.*
(ii) *Suppose, moreover, that $y_0 = (1-\beta_0)u + \beta_0 x_0$. If one of the sequences (x_n), (y_n) is (T-)asymptotically regular with rate Φ, then the other one is (T-)asymptotically regular with rate Φ'.*

Proof. (i) Let $k \in \mathbb{N}$ and $n \geq \Phi'(k)$. Assume first that Φ is a rate of T-asymptotic regularity of (u_n). We get that

$$\begin{aligned} d(x_n,Tx_n) &\leq d(x_n,u_n) + d(u_n,Tu_n) + d(Tu_n,Tx_n)\\ &\leq 2d(x_n,u_n) + d(u_n,Tu_n) \quad \text{[since } T \text{ is nonexpansive]}\\ &\leq 2d(x_n,u_n) + \frac{1}{3(k+1)} \quad \text{[since } n \geq \Phi(3k+2)]\\ &\leq \frac{2}{3(k+1)} + \frac{1}{3(k+1)} = \frac{1}{k+1}, \end{aligned}$$

as $n \geq \alpha(3k+2)$, so we can apply Lemma 4.1.

Assume now that Φ is a rate of asymptotic regularity of (u_n). Then

$$\begin{aligned} d(x_n,x_{n+1}) &\leq d(x_n,u_n) + d(u_n,u_{n+1}) + d(u_{n+1},x_{n+1})\\ &\leq d(x_n,u_n) + d(u_{n+1},x_{n+1}) + \frac{1}{3(k+1)}\\ &\leq \frac{2}{3(k+1)} + \frac{1}{3(k+1)} = \frac{1}{k+1}. \end{aligned}$$

The proof for the case when Φ is a rate of (T-)asymptotic regularity of (x_n) follows by symmetry.

(ii) We have, by Proposition 3.2, that $y_n = u_n$ for all $n \in \mathbb{N}$. Apply (i).

□

It follows that if the starting points $x_0, y_0 \in C$ satisfy $y_0 = (1-\beta_0)u + \beta_0 x_0$, then (x_n) is (T-)asymptotically regular if and only if (y_n) is (T-)asymptotically regular.

4.1. **On rates of (T-)asymptotic regularity of the modified Halpern iteration.** In [28, Propositions 6.1, 6.2], Schade and the second author computed uniform rates of (T-)asymptotic regularity of the modified Halpern iteration in W-hyperbolic spaces. The hypotheses on the sequences (λ_n), (β_n) used in [28] were $(H1_q)$ (or - for strictly positive $\beta_n > 0$ - equivalently, $(H1^*_q)$), $(H2_q)$ - $(H4_q)$ and $(H6_q)$. We improve these results by showing that the hypothesis $(H6_q)$ can be weakened to $(H5_q)$.

Let (y_n) be the modified Halpern iteration, given by (4). Let $M \in \mathbb{N}^*$ be such that

$$M \geq 4\max\{d(u,p), d(y_0,p)\} \tag{12}$$

for some $p \in Fix(T)$.

The following lemma collects some properties of (y_n) that will be useful in the sequel.

Lemma 4.3. [9, 28] *For all $n \geq 1$,*

$$d(y_n,u) \leq M, d(Ty_n,u) \leq M \text{ and } d(y_n,y_{n+1}) \leq M, \tag{13}$$

$$d(y_n,Ty_n) \leq d(y_n,y_{n+1}) + (1-\beta_{n+1})d(u,Ty_n) + \beta_{n+1}(1-\lambda_n)d(y_n,Ty_n), \tag{14}$$

$$d(y_{n+1},y_n) \leq \beta_{n+1}\big(d(y_n,y_{n-1}) + |\lambda_n - \lambda_{n-1}|d(y_{n-1},Ty_{n-1})\big) + |\beta_{n+1}-\beta_n|c_n, \tag{15}$$

where $c_n = (1-\lambda_{n-1})d(u,y_{n-1}) + \lambda_{n-1}d(u,Ty_{n-1})$.

Proof. These properties are proved in [28] following [9] which treats the case of CAT(0) spaces. As we pointed out in Remark 3.1, the sequence (x_n) and the scalars β_n, α_n in [28] correspond to (y_n) and $1-\beta_{n+1}, 1-\lambda_n$ in our current paper. Then (13) is [28, Lemma 5.2(6),(8),(9)], (14) is [28, Lemma 5.1(6)], and (15) is [28, Proof of Lemma 5.1(3), last line on p.10]. □

Proposition 4.4. *Assume that $(H2_q)$, $(H3_q)$ hold. Define*

$$\gamma(k) = \max\left\{\sigma_2(8M(k+1)-1), \sigma_3(4M(k+1)-1)\right\}. \tag{16}$$

(i) *If $(H1_q)$ holds, then (y_n) is asymptotically regular with rate*

$$\Sigma(k) = \sigma_1\big(\gamma(k) + \lceil \ln(M(k+1)) \rceil + 1\big) + 1. \tag{17}$$

(ii) *Suppose that $(H1^*_q)$ holds, and $\psi : \mathbb{N} \to \mathbb{N}^*$ satisfies*

$$\frac{1}{\psi(k)} \leq \prod_{n=0}^{\gamma(k)} \beta_{n+1}. \tag{18}$$

Then (y_n) is asymptotically regular with rate

$$\Sigma^*(k) = \sigma_1^*\left(M\psi(k)(k+1)-1\right) + 1. \tag{19}$$

Proof. This result is proven - in a different notation - in [28, Propositions 6.1, 6.2] observing that only the assumptions stated above are used there, where [28] in turn is based on [18]. More specifically:

(i) replace the notations used for the modified Halpern iteration in [28] with the ones from this paper (see Remark 3.1).
(ii) use $\frac{1}{k+1}$ instead of ε.
(iii) replace M_2 with M, ψ_α with σ_3, ψ_β with σ_2, and θ_β with σ_1 in (i) and with σ_1^* in (ii), D with $\frac{1}{\psi(k)}$ in (ii), with the corresponding changes in the parameters, due to the definitions, used in this paper, of the rates as mappings $\mathbb{N} \to \mathbb{N}$.

We get that Σ from (i) is $\tilde{\Phi}$ from [28, Proposition 6.1] with $\psi_\beta\left(\frac{\varepsilon}{8M_2}\right)$ replaced by $\sigma_2(8M(k+1)-1)$, $\psi_\alpha\left(\frac{\varepsilon}{4M_2}\right)$ replaced by $\sigma_3(4M(k+1)-1)$, and $\left\lceil\frac{M_2}{\varepsilon}\right\rceil$ replaced by $M(k+1)$. Furthermore, Σ^* from (ii) is $\tilde{\Phi}$ from [28, Proposition 6.2] with $\theta_\beta\left(\frac{D\varepsilon}{M_2}\right)$ replaced by $\sigma_1^*\left(M\psi(k)(k+1)-1\right)$. $\square$

Proposition 4.5. *Assume that $(H4_q)$ and $(H5_q)$ hold. If $\Sigma:\mathbb{N}\to\mathbb{N}$ is a rate of asymptotic regularity of (y_n), then*

$$\widehat{\Sigma}(k)=\max\{N_\Lambda,\Sigma(2\Lambda(k+1)-1),\sigma_4(2M\Lambda(k+1)-1)\} \tag{20}$$

is a rate of T-asymptotic regularity of (y_n).

Proof. We get that for all $n\in\mathbb{N}$,

$$\begin{aligned}
d(y_n,Ty_n) &\overset{(14)}{\leq} d(y_n,y_{n+1})+(1-\beta_{n+1})d(u,Ty_n)+\beta_{n+1}(1-\lambda_n)d(y_n,Ty_n)\\
&\overset{(13)}{\leq} d(y_n,y_{n+1})+M(1-\beta_{n+1})+(1-\lambda_n)d(y_n,Ty_n).
\end{aligned}$$

After moving $(1-\lambda_n)d(y_n,Ty_n)$ to the left-hand side, we get that, for all $n\in\mathbb{N}$,

$$\lambda_n d(y_n,Ty_n)\leq d(y_n,y_{n+1})+M(1-\beta_{n+1}). \tag{21}$$

Let now $n\geq\widehat{\Sigma}(k)$. Since $n\geq N_\Lambda$, we can apply $(H5_q)$ to obtain that $\lambda_n\geq\frac{1}{\Lambda}$. It follows from (21) that

$$\frac{1}{\Lambda}d(y_n,Ty_n)\leq d(y_n,y_{n+1})+M(1-\beta_{n+1}),$$

hence

$$d(y_n,Ty_n)\leq\Lambda d(y_n,y_{n+1})+M\Lambda(1-\beta_{n+1}). \tag{22}$$

As $n\geq\Sigma(2\Lambda(k+1)-1)$, we have that

$$d(y_n,y_{n+1})\leq\frac{1}{2\Lambda(k+1)}. \tag{23}$$

Since $n\geq\sigma_4(2M\Lambda(k+1)-1)$, we get that

$$1-\beta_{n+1}\leq\frac{1}{2M\Lambda(k+1)}. \tag{24}$$

Apply (22), (23) and (24) to conclude that

$$d(y_n,Ty_n)\leq\frac{1}{2(k+1)}+\frac{1}{2(k+1)}=\frac{1}{k+1}.$$

$\square$

Thus, as an application of Propositions 4.4, 4.5, one computes also rates of T-asymptotic regularity of the modified Halpern iteration.

One can easily see that particularizing the rates obtained in Propositions 4.4.(ii) and 4.5 to the scalars $\lambda_n=\lambda\in(0,1]$ and $\beta_n=1-\frac{1}{n+1}$ yields to quadratic rates of (T-)asymptotic regularity. In [27] a linear rate of convergence is obtained for some other Halpern-type iteration in the normed case for $\beta_n=1-\frac{2}{n}$. We now show that we also obtain this for the modified Halpern iteration in W-hyperbolic spaces using [27, Lemma 3]:

Lemma 4.6 ([27])**.** *Let $L > 0$ and (a_n) be a sequence of non-negative real numbers with $a_1 \le L$ such that for $b_n = \min\{2/n, 1\}$ we have for all $n \ge 1$,*

$$a_{n+1} \le (1 - b_{n+1})a_n + (b_n - b_{n+1})c_n,$$

where (c_n) is a sequence of reals such that $c_n \le L$ for all $n \ge 1$.
Then $a_n \le 2L/n$ for all $n \ge 1$.

Proposition 4.7. *Let $\lambda_n = \lambda \in (0,1]$, $\beta_1 = 0$ and $\beta_n = 1 - \frac{2}{n}$ for $n \ge 2$. Then for all $n \ge 0$,*

$$d(y_{n+1}, y_n) \le \frac{2M}{n+1},$$
$$d(y_n, Ty_n) \le \frac{4M}{\lambda(n+1)},$$

where $M \in \mathbb{N}^*$ is such that $M \ge 4\max\{d(u,p), d(y_0,p)\}$.

Proof. Apply (15) to get that for all $n \ge 1$,

$$d(y_{n+1}, y_n) \le \beta_{n+1} d(y_n, y_{n-1}) + |\beta_{n+1} - \beta_n| c_n,$$

where $c_n = (1-\lambda)d(u, y_{n-1}) + \lambda d(u, Ty_{n-1})$. By (13), we have that

$$\max\{d(y_1, y_0), c_n\} \le M.$$

As $\beta_n = 1 - \min\{2/n, 1\}$ for all $n \ge 1$, we can apply Lemma 4.6 with $a_n = d(y_n, y_{n-1})$ to obtain that for $n \ge 0$,

$$d(y_{n+1}, y_n) \le \frac{2M}{n+1}$$

and so by - the proof of - (22) above

$$d(y_n, Ty_n) \le \frac{2M}{\lambda(n+1)} + \frac{2M}{\lambda(n+1)} = \frac{4M}{\lambda(n+1)}.$$

□

4.2. **From modified Halpern iteration to Tikhonov-Mann iteration.** We derive rates of (T-)asymptotic regularity of the Tikhonov-Mann iteration from the rates of the modified Halpern iterations computed in Subsection 4.1.
Let (x_n) be the Tikhonov-Mann iteration, defined by (4), and $K \in \mathbb{N}^*$ be such that $K \ge M_p$, where p is a fixed point of T and M_p is given by (8).

Proposition 4.8. *Assume that $(H2_q)$, $(H3_q)$, and $(H4_q)$ hold.*

(i) *If $(H1_q)$ holds and Σ is defined as in Proposition 4.4.(i), then (x_n) is asymptotically regular with rate*

$$\Phi(k) = \max\{\sigma_4(6K(k+1) - 1), \Sigma(3k+2)\}.$$

(ii) *If $(H1^*_q)$ holds and ψ, Σ^* are as in Proposition 4.4.(ii), then (x_n) is asymptotically regular with rate*

$$\Phi(k) = \max\{\sigma_4(6K(k+1) - 1), \Sigma^*(3k+2)\}.$$

(iii) *If $(H5_q)$ holds and Σ is a rate of asymptotic regularity of (y_n), then*

$$\widehat{\Phi}(k) = \max\{\sigma_4(6K(k+1) - 1), N_\Lambda, \Sigma(6\Lambda(k+1) - 1), \sigma_4(24K\Lambda(k+1) - 1)\}$$

is a rate of T-asymptotic regularity of (x_n).

Proof. Define $y_0 = (1-\beta_0)u + \beta_0 x_0$ and consider the modified Halpern iteration (y_n) starting with y_0. By an application of (W1), we get that

$$d(y_0, p) \leq (1-\beta_0)d(u,p) + \beta_0 d(x_0, p) \leq K.$$

Hence, we can use Proposition 4.4 with $M = 4K$ to get rates of asymptotic regularity of (y_n). Apply now Proposition 4.2.(ii) to obtain (i) and (ii).
As for item (iii), assume, furthermore, that $(H5_q)$ holds and let Σ be a rate of asymptotic regularity of (y_n). Then $\widehat{\Sigma}$ defined as in Proposition 4.5 is a rate of T-asymptotic regularity of (y_n). Applying Proposition 4.2.(ii), we get that

$$\widehat{\Phi}(k) = \max\left\{\sigma_4(6K(k+1)-1), N_\Lambda, \Sigma(6\Lambda(k+1)-1), \sigma_4(24K\Lambda(k+1)-1)\right\}$$

is a rate of T-asymptotic regularity of (x_n). □

We consider now again the case $\lambda_n = \lambda \in (0,1]$, $\beta_1 = 0$ and $\beta_n = 1 - \frac{2}{n}$ for $n \geq 2$. Let $y_0 = (1-\beta_0)u + \beta_0 x_0$. As $y_n = u_n$ (by Proposition 3.2), we get from (9) that for all $n \geq 1$,

$$d(x_n, y_n) \leq \frac{4K}{n}.$$

Since $d(y_0, p) \leq K$, we can apply Proposition 4.7 with $M = 4K$ and reason as in the proof of Proposition 4.2.(i) to obtain that for all $n \geq 1$,

$$d(x_n, x_{n+1}) \leq d(x_n, y_n) + d(x_{n+1}, y_{n+1}) + d(y_n, y_{n+1}) \leq \frac{4K}{n} + \frac{12K}{n+1} < \frac{16K}{n},$$

$$d(x_n, Tx_n) \leq 2d(x_n, y_n) + d(y_n, Ty_n) \leq \frac{8K}{n} + \frac{16K}{\lambda(n+1)} < \frac{24K}{\lambda n}.$$

So we have obtained a linear rate of asymptotic regularity also for the Tikhonov-Mann iteration as a consequence of the corresponding fact for the modified Halpern iteration and the reduction of the former to the latter.

4.3. **From Tikhonov-Mann iteration to modified Halpern iteration.** Let (y_n) be the modified Halpern iteration, defined by (5), p be a fixed point of T, and $K \in \mathbb{N}^*$ be such that $K \geq \max\{d(u,p), d(y_0,p)\}$.
The following proposition gives rates of (T-)asymptotic regularity of (y_n).

Proposition 4.9. *Assume that $(H2_q)$, $(H3_q)$, and $(H4_q)$ hold. Define*

$$\chi(k) = \max\{\sigma_2(8K(k+1)-1), \sigma_3(8K(k+1)-1)\},$$
$$\theta(k) = \sigma_4(6K(k+1)-1).$$

(i) *Suppose that $(H1_q)$ holds. Then*
 (a) *(y_n) is asymptotically regular with rate*

 $$\Sigma(k) = \max\{\theta(k), \sigma_1(\chi(9k+8) + 2 + \lceil \ln(18K(k+1)) \rceil) + 1\}.$$

 (b) *If $(H5_q)$ holds, then (y_n) is T-asymptotically regular with rate*

 $$\widehat{\Sigma}(k) = \max\{\theta(k), N_\Lambda, \Sigma(6\Lambda(k+1)-1), \sigma_4(12K\Lambda(k+1)-1)\}.$$

(ii) *Suppose that $(H1_q^*)$ holds and that $\psi : \mathbb{N} \to \mathbb{N}^*$ is such that $\frac{1}{\psi(k)} \leq \prod_{n=0}^{\chi(3k+2)} \beta_{n+1}$. Then*

(a) (y_n) *is asymptotically regular with rate*

$$\Sigma^*(k) = \max\{\theta(k), \sigma_1^*(\psi^*(k)-1)+1, \chi(9k+8)+2\},$$

where $\psi^*(k) = 18K(k+1)\psi(3k+2)$.

(b) *If* $(H5_q)$ *holds, then* (y_n) *is* T*-asymptotically regular with rate*

$$\widehat{\Sigma^*}(k) = \max\{\theta(k), N_\Lambda, \Sigma^*(6\Lambda(k+1)-1), \sigma_4(12K\Lambda(k+1)-1)\}.$$

Proof. Take $\beta_0 = 1$. Then, by (W5), $y_0 = (1-\beta_0)u + \beta_0 y_0$. Apply [7, Theorems 4.1,4.2] for the Tikhonov-Mann iteration (x_n) starting with y_0 to obtain rates of (T-)asymptotic regularity for this iteration and use Proposition 4.2.(ii) to translate them into rates for (y_n).
For the proof of (ii), remark that if $(H1_q^*)$ holds, then σ_1^* is also a rate of convergence of $\left(\prod_{n=0}^{\infty} \beta_{n+1}\right)$ towards 0, hence [7, $(C2_q)$] holds with $\sigma_2 := \sigma_1^*$. □

The first and the third author computed, for the particular case $\lambda_n = \lambda \in (0,1]$ and $\beta_n = 1 - \frac{1}{n+1}$, quadratic rates of ($T$-)asymptotic regularity for the Tikhonov-Mann iteration (see [7, Corollary 4.3]). We show in the sequel that we can use Lemma 4.6 for this iteration, too, and obtain, as a consequence, linear rates of (T-)asymptotic regularity by letting $\beta_n = 1 - \frac{2}{n}$.

Proposition 4.10. *Assume that* $\lambda_n = \lambda \in (0,1]$, $\beta_1 = 0$ *and* $\beta_n = 1 - \frac{2}{n}$ *for* $n \geq 2$, *and let* (x_n) *be the Tikhonov-Mann iteration. Then for all* $n \geq 1$,

$$d(x_{n+1}, x_n) \leq \frac{4K}{n},$$

$$d(x_n, Tx_n) \leq \frac{8K}{\lambda n}.$$

Proof. Using [7, Proposition 3.2(7)], we get that for all $n \geq 1$,

$$d(x_{n+2}, x_{n+1}) \leq \beta_{n+1} d(x_{n+1}, x_n) + 2K|\beta_{n+1} - \beta_n| \leq \beta_{n+1} d(x_{n+1}, x_n) + 2K.$$

Moreover, $d(x_1, x_0) \leq d(x_1, p) + d(p, x_0) \leq 2K$. Applying Lemma 4.6 for $a_n = d(x_{n+1}, x_n)$, $b_n = 1 - \beta_n$, $c_n = 2K$, and $L = 2K$, we get that for all $n \geq 1$,

$$d(x_{n+1}, x_n) \leq \frac{4K}{n}.$$

We obtain, as in the proof of [7, Proposition 5.5], that for all $n \geq 1$,

$$d(x_n, Tx_n) \leq \frac{1}{\lambda} d(x_n, x_{n+1}) + \frac{2K}{\lambda}(1-\beta_n) \leq \frac{4K}{\lambda n} + \frac{4K}{\lambda n} = \frac{8K}{\lambda n}.$$

□

We argue now as in Section 4.2 to get, from Proposition 4.10, linear rates for the modified Halpern iteration (y_n): for all $n \geq 1$,

$$d(y_{n+1}, y_n) \leq d(x_n, y_n) + d(x_{n+1}, y_{n+1}) + d(x_n, x_{n+1}) \leq \frac{12K}{n},$$

$$d(y_n, Ty_n) \leq 2d(x_n, y_n) + d(x_n, Tx_n) \leq \frac{16K}{\lambda n}.$$

So for $d(y_n, Ty_n)$ the detour through the Tikhonov-Mann iteration gives (almost) exactly the same rate as the direct approach in Proposition 4.7 while the latter gives the slighty better constant '8' in the rate for $d(y_{n+1}, y_n)$.

5. Rates of metastability

Recall that if X is a metric space and (a_n) is a sequence in X, a function $\Omega : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is a rate of metastability of (a_n) if it satisfies the following: for all $k \in \mathbb{N}$ and all $g : \mathbb{N} \to \mathbb{N}$, there exists $N \leq \Omega(k, g)$ such that

$$\forall i, j \in [N, N + g(N)] \left(d(a_i, a_j) \leq \frac{1}{k+1} \right).$$

Noneffectively, the above statement of metastability is trivially equivalent to the Cauchy property of (a_n) and hence to its convergence if X is complete. Whereas there are no computable rates of convergence for the iterations we consider (as a consequence of [25]), effective rates of metastability can be extracted even from highly noneffective convergence proofs (as the one given in [9] using Banach limits and hence the axiom of choice) by general tools from proof theory. See [15] as well as the recent survey [17], where also a short history of metastability is given which goes back to Kreisel's seminal work in the early 50's ([20, 21]), while the term 'metastability' was coined by Tao [30] who in turn refers to Jennifer Chayes' concept of a 'metastability principle'.

The following result shows that, as in the case of (T-)asymptotic regularity, there is a strong relation between rates of metastability of the Tikhonov-Mann iteration (x_n) and the ones of the modified Halpern iteration (y_n). The setting is the same as in Section 4.

Proposition 5.1. *Assume that $(H4_q)$ holds and let $\Omega : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$. Define $\Omega' : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ by*

$$\Omega'(k, g) = \widetilde{\Omega}(3k + 2, g, \alpha(3k + 2)), \tag{25}$$

where α is given by (10) and $\widetilde{\Omega} : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ is defined by

$$\widetilde{\Omega}(k, g, q) = \Omega(k, g_q) + q,$$

with $g_q : \mathbb{N} \to \mathbb{N}$, $g_q(n) = g(n + q) + q$.

(i) *If Ω is a rate of metastability of one of the sequences (x_n), (u_n), then Ω' is a rate of metastability of the other one.*

(ii) *Suppose that $y_0 = (1 - \beta_0)u + \beta_0 x_0$. If one of the sequences (x_n), (y_n) is Cauchy with rate of metastability Ω, then the other one is Cauchy with rate of metastability Ω'.*

Proof. (i) Assume first that Ω is a rate of metastability of (u_n).

Claim: For all $k \in \mathbb{N}$, $g : \mathbb{N} \to \mathbb{N}$, $q \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that

$$q \leq N \leq \widetilde{\Omega}(k, g, q) \text{ and } \forall i, j \in [N, N + g(N)] \left(d(u_i, u_j) \leq \frac{1}{k+1} \right).$$

Proof of claim: Let $k \in \mathbb{N}$, $g : \mathbb{N} \to \mathbb{N}$, $q \in \mathbb{N}$. As Ω is a rate of metastability of (u_n), it follows that there exists $N_0 \leq \Omega(k, g_q)$ such that

$$\forall i, j \in [N_0, N_0 + g_q(N_0)] \left(d(u_i, u_j) \leq \frac{1}{k+1} \right).$$

Let $N := N_0 + q$. Then $q \leq N \leq \Omega(k, g_q) + q = \widetilde{\Omega}(k, g, q)$. Furthermore, $[N, N + g(N)] = [N_0 + q, N_0 + q + g(N_0 + q)] = [N_0 + q, N_0 + g_q(N_0)] \subseteq [N_0, N_0 + g_q(N_0)]$, hence $d(u_i, u_j) \leq \frac{1}{k+1}$ for all $i, j \in [N, N + g(N)]$. ■

Let $k \in \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$. Apply the claim for $k := 3k+2$, g and $q := \alpha(3k+2)$ to get the existence of $N \in \mathbb{N}$ such that $\alpha(3k+2) \leq N \leq \widetilde{\Omega}(3k+2, g, \alpha(3k+2)) = \Omega'(k, g)$ such that

$$\forall i, j \in [N, N+g(N)] \left(d(u_i, u_j) \leq \frac{1}{3(k+1)} \right). \tag{26}$$

It follows that for all $i, j \in [N, N+g(N)]$,

$$\begin{aligned} d(x_i, x_j) &\leq d(x_i, u_i) + d(u_i, u_j) + d(u_j, x_j) \\ &\leq \frac{1}{3(k+1)} + d(x_i, u_i) + d(u_j, x_j) \quad \text{by (26)} \\ &\leq \frac{1}{3(k+1)} + \frac{1}{3(k+1)} + \frac{1}{3(k+1)} = \frac{1}{k+1}, \end{aligned}$$

as $i, j \geq N \geq \alpha(3k+2)$, and α is a rate of convergence towards 0 of $(d(x_n, u_n))$, by Lemma 4.1.

The proof for the case when Ω is a rate of metastability of (x_n) is similar.

(ii) Apply Proposition 3.2 and (i).

□

The main results of [28] are quantitative versions of the strong convergence, proved in [9], of the modified Halpern iteration (y_n) in complete CAT(0) spaces. These quantitative versions provide effective uniform rates of metastability for (y_n) (see [28, Theorems 4.1, 4.2] and also note [19] for a numerical improvement). In Subsection 4.1 we improved the quantitative results on the asymptotic regularity of (y_n) obtained in [28, Propositions 6.1, 6.2] by weakening the hypothesis $(H6_q)$ to $(H5_q)$. One can easily see that this also eliminates the hypothesis $(H6_q)$ in favor of $(H5_q)$ also in [28, Theorems 4.1, 4.2], as it is not used in their proofs except via the rate of asymptotic regularity. Hence, for CAT(0) spaces, new rates of metastability for (y_n) are obtained by considering the ones from [28] with the new rates of (T-)asymptotic regularity computed in Subsection 4.1. By Proposition 5.1.(ii), it follows that we can compute rates of metastability for the Tikhonov-Mann iteration (x_n) in CAT(0) spaces, assuming that $(H1_q)$ (or, equivalently for $\beta_n > 0$, $(H1_q^*)$) and $(H2_q)$-$(H5_q)$ hold.

6. Conclusions

In this paper we showed that there is a strong relation between the modified Halpern iteration and the Tikhonov-Mann iteration for nonexpansive mappings. Thus, asymptotic regularity and strong convergence results can be translated from one iteration to the other. This translation holds also for quantitative versions of these results, providing rates of asymptotic regularity and rates of metastability. A future direction of research is to explore similar connections to other modified versions of the Halpern and Mann iterations. One candidate is the alternating Halpern-Mann iteration, introduced recently by Dinis and Pinto [11].

By applying a lemma on sequences of real numbers due to Sabach and Shtern [27, Lemma 3], we obtained, for a particular choice of the scalars, linear rates of asymptotic regularity for the modified Halpern and Tikhonov-Mann iterations. Previous results guaranteed only quadratic such rates. This lemma was applied for the first time in [27] to get linear rates of asymptotic regularity for the sequential averaging method (SAM), developed in [33]. As a consequence, one gets linear rates for the Halpern iteration also obtained (with an optimal constant) in the case of Hilbert spaces using a different technique in [24]. Leuştean and Pinto [22] computed, using the same method, linear rates for the alternating Halpern-Mann iteration. Recently, the first and the third author in [8] applied (a version of) [27, Lemma 3] to other classes of nonlinear iterations and, as a result, obtain linear rates of asymptotic regularity for these iterations.

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